Classification of matrix systems of $P_2 - P_6$ type with Okamoto integral

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Introduction

The Okamoto polynomial Hamiltonian of the sixth Painlevé equation is given by:

$$z(z-1)H = u^3v^2 - u^2v^2 - \kappa_1 u^2 v + \kappa_2 uv - \kappa_3 u +$$

$$z\left(-u^2v^2 + uv^2 + \kappa_4 uv + (\kappa_1 - \kappa_2 - \kappa_4)v\right),$$

where k_i are arbitrary constants, $z \in \mathbb{C}$.

The corresponding Hamiltonian system is

$$\begin{cases} z(z-1)\dfrac{du}{dz} = 2u^3v - 2u^2v - \kappa_1 u^2 + \kappa_2 u \\ \qquad +z\left(-2u^2v + 2uv + \kappa_4 u + \kappa_1 - \kappa_2 - \kappa_4\right), \\ z(z-1)\dfrac{dv}{dz} = -3u^2v^2 + 2uv^2 + 2\kappa_1 uv - \kappa_2 v + \kappa_3 \\ \qquad +z\left(2uv^2 - v^2 - \kappa_4 v\right), \end{cases} \tag{1}$$

The variable u satisfies the Painlevé-6 equation.

The system has the form

$$\begin{cases} f(z)\dfrac{du}{dz} &= P_1(u,v) + z\,Q_1(u,v), \\ f(z)\dfrac{dv}{dz} &= P_2(u,v) + z\,Q_2(u,v), \end{cases} \tag{2}$$

while the Hamiltonian has the following structure $f(z)H = H_1 + z\,H_2$, where P_i, Q_i, H_i are polynomials in u, v. All other scalar Painlevé systems and their Hamiltonians have the same structure.

Let us consider the system

$$\begin{cases} \dfrac{du}{dt} &= P_1(u,v) + z\,Q_1(u,v), \\ \dfrac{dv}{dt} &= P_2(u,v) + z\,Q_2(u,v), \end{cases} \tag{3}$$

where we regard z as a parameter. We call (3) *auxiliary autonomous system* for (2).

It follows from the fact that (1) is a Hamiltonian system with the Hamiltonian H that

$$J = H_1 + z\,H_2, \tag{4}$$

where

$$H_1 = u^3v^2 - u^2v^2 - \kappa_1 u^2 v + \kappa_2 uv - \kappa_3 u,$$

$$H_2 = -u^2v^2 + uv^2 + \kappa_4 uv + (\kappa_1 - \kappa_2 - \kappa_4)v,$$

is an integral of motion for system (3) i.e. $\frac{dJ}{dt} = 0$. We call the function J the *Okamoto integral.*

For any N the system

$$\begin{cases} u_\tau &= J^N\,u_t, \\ v_\tau &= J^N\,v_t \end{cases} \tag{5}$$

is an infinitesimal symmetry.

Consider systems of the form (3), where P_i and Q_i are non-commutative polynomials given by

$$P_1(u,v) = a_1u^3v + a_2u^2vu + a_3uvu^2 + (2-a_1-a_2-a_3)vu^3$$
$$+c_1u^2v + (-2-c_1-c_2)uvu + c_2vu^2 - \kappa_1u^2 + \kappa_2u,$$

$$Q_1(u,v) = f_1u^2v + (-2-f_1-f_2)uvu + f_2vu^2$$
$$+h_1uv + (2-h_1)vu + \kappa_4u + (\kappa_1-\kappa_2-\kappa_4),$$

and

$$P_2(u,v) = b_1u^2v^2 + b_2uvuv + b_3uv^2u + b_4vu^2v + b_5vuvu$$
$$+(-3-\sum b_i)v^2u^2 + d_1uv^2 + (2-d_1-d_2)vuv$$
$$+d_2v^2u + e_1vu + (2\kappa_1-e_1)uv - \kappa_2v + \kappa_3,$$

$$Q_2(u,v) = g_1uv^2 + (2-g_1-g_2)vuv + g_2v^2u - v^2 - \kappa_4v.$$

We assume that all coefficients are complex constants. If $f(z) = z(z-1)$, then the corresponding system (2) is a natural non-commutative generalization of the Painlevé-6 system.

We postulate the existence of non-abelian integral of motion of the form

$$J = H_1(u,v) + z\, H_2(u,v), \tag{6}$$

where

$$\begin{aligned} H_1(u,v) = {} & p_1 u^3 v^2 + p_2 u^2 v u v + p_3 u^2 v^2 u + p_4 u v u^2 v + p_5 u v u v u \\ & + p_6 u v^2 u^2 + p_7 v u^3 v + p_8 v u^2 v u + p_9 v u v u^2 \\ & + \left(1 - \sum p_i\right) v^2 u^3 + q_1 u^2 v^2 + q_2 u v u v + q_3 u v^2 u + q_4 v u^2 v \\ & + q_5 v u v u + \left(-1 - \sum q_i\right) v^2 u^2 + r_1 u^2 v + r_2 u v u \\ & + (-\kappa_1 - \sum r_i) v u^2 + s_1 u v + (\kappa_2 - s_1) v u - \kappa_3 u, \end{aligned}$$

$$\begin{aligned} H_2(u,v) = {} & t_1 u^2 v^2 + t_2 u v u v + t_3 u v^2 u + t_4 v u^2 v + t_5 v u v u \\ & + (-1 - \sum t_i) v^2 u^2 + x_1 u v^2 + x_2 v u v + (1 - \sum x_i) v^2 u \\ & + y_1 u v + (\kappa_4 - y_1) v u + (\kappa_1 - \kappa_2 - \kappa_4) v \end{aligned}$$

As a result, we found 18 non-abelian systems (2) of Painlevé type. A transformation group acts on the set of these system. There are 3 orbits of the group action and three non-equivalent systems corresponding to these orbits.

All these systems are not Hamiltonian and therefore our approach cannot reconstruct the known non-abelian Hamiltonian P_6 system. However, we obtain an interesting class of integrable P_6 systems.

To justify their integrability, we find the isomonodromic Lax representations of the form

$$\mathcal{A}_z - \mathcal{B}_\lambda = [\mathcal{B}, \mathcal{A}] \tag{7}$$

for these systems.

Non-abelian ODEs

The systems have the form

$$\frac{dx_\alpha}{dt} = F_\alpha(\mathbf{x}), \qquad \mathbf{x} = (x_1, ..., x_N), \tag{8}$$

where $x_1, \ldots, x_N$ are generators of the free unital associative algebra $\mathcal{A}$ over $\mathbb{C}$. Actually, (8) is a notation for the derivation d_t of $\mathcal{A}$ such that $d_t(x_i) = F_i$. The element $d_t(z)$ is uniquely determined for any element $z \in \mathcal{A}$ by the Leibniz identity.

Usually, the first integrals of a system (8) are some elements of the quotient vector space $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$, which is a formalisation of integrals of the form $trace(f(x_1, ..., x_N))$ in the matrix case $x_i(t) \in Mat_m$. For the Hamiltonian non-abelian systems the Hamiltonians are first integrals of this kind. However in this paper we are dealing with the first integrals that are elements of $\mathcal{A}$.

Definition An element $j \in \mathcal{A}$ are called a first integral of (8) iff $d_t(j) = 0$.

For non-abelian systems with two variables u and v a special integral $I = uv - vu$ appears in the following statement:

Lemma A system is Hamiltonian with respect to the canonical symplectic structure i.e. has the form

$$\begin{cases} \dfrac{du}{dt} & = & \dfrac{\partial H}{\partial v}, \\ \dfrac{dv}{dt} & = & -\dfrac{\partial H}{\partial u}, \end{cases} \tag{9}$$

where $H \in \mathcal{A}$ and $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial v}$ are non-abelian derivatives iff the system has the first integral I.

Non-abelian partial derivatives $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ for arbitrary polynimial $f \in \mathcal{A}$ are defined by the identity

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n,$$

where the additional non-abelian variables dx_i are supposed to be moved to the right by the cyclic permutations of generators in monomials. Notice that $\frac{\partial}{\partial x_i}$ themselves are not vector fields in the nonabelian case.

Example Let $f = u^2vuv$. We have
$df = du\, uvuv + u\, du\, vuv + u^2\, dv\, uv + u^2v\, du\, v + u^2vu\, dv.$
Now we make cyclic permutations in monomials to bring all du, dv to the end in each monomial. We obtain
$uvuv\, du + vuvu\, du + uvu^2\, dv + vu^2v\, du + u^2vu\, dv$ and, therefore,

$$\frac{\partial f}{\partial u} = uvuv + vuvu + vu^2v, \quad \frac{\partial f}{\partial v} = uvu^2 + u^2vu$$

.

Notice that the Hamiltonian H of a system (9) is not a first integral in the sense od our definition.

We assume that the auxiliary system (3) has an Okamoto first integral of the form (6). In both the system and in the integral the variable z plays the role of arbitrary parameter. Using the terminology of the bi-Hamiltonian formalism, we have pencils of two non-abelian dynamical systems and two non-abelian first integrals.

Instead of algebra $\mathcal{A}$ with multiplication xy one can consider the associative algebra with the opposite product $x \star y = yx$. The transition to the opposite multiplication is represented by the involution $\tau : \mathcal{A} \mapsto \mathcal{A}$ defined by

$$\tau(x_i) = x_i, \quad \tau(ax + by) = a\tau(x) + b\tau(y), \quad \tau(xy) = \tau(y)\tau(x),$$

where $x, y \in \mathcal{A},\ a, b \in \mathbb{C}$. This involution is called *transposition.*

All properties of integrable systems such as the existence of first integrals, infinitesimal symmetries, Lax representations etc. are invariant with respect to τ.

P_6 systems

Differentiating the integral with respect to the system, we obtain a polynomial $P(u, v, z)$ of degree 8. The simplest equations from this system are:

$$p_1 = p_3 = p_6 = p_7 = 0, \qquad p_2 = 1 - p_4 - p_5 - p_8 - p_9.$$

In turns out that all coefficients of polynomials P_i, Q_i can be expressed in terms of the Okamoto integral:

$$a_1 = 1 - p_4 - p_5 - p_8 - p_9, \qquad a_2 = 1 + p_4 - p_8 - p_9,$$

$$a_3 = p_5 + 2p_8 + p_9, \qquad b_1 = -1 + p_4 + p_5 + p_8 + p_9,$$

$$b_2 = -2 + p_5 + 2p_8 + 2p_9, \qquad b_3 = 0, \qquad b_4 = -p_4 - p_5 - p_8,$$

$$b_5 = -p_5 - 2p_8 - 2p_9, \qquad c_1 = -d_1 = 2q_1 + q_2,$$

$$c_2 = -d_2 = -2 - 2q_1 - 2q_2 - 2q_3 - 2q_4 - q_5, \qquad h_1 = 2x_1 + x_2,$$

$$e_1 = 2\kappa_1 + 2r_1 + r_2,$$

$$f_2 = 1 - t_3 + x_1 + x_2 - p_5 - p_8 - 2p_9 + 2q_1 + 2q_2 + q_3 + 2q_4 + q_5,$$

$$f_1 = -2 - t_3 - x_1 + p_4 + p_5 + 2p_8 + 2p_9 - 2q_1 - q_2 - q_3;$$

Equating to zero the coefficients of different monomials of degree 8 in $P(u, v, z)$, we arrive at a system of nonlinear algebraic equations for the variables a_i, $i = 1, 2, 3$, b_i, $i = 1, ..., 5$ and p_i, $i = 1, ..., 9$.

Using the above formulas, we can eliminate a_i, b_i and p_1, p_2, p_3, p_6, p_7 to obtain a system for p_4, p_5, p_8, p_9, which is equivalent to

$$(p_4 - 1)p_4 = (p_5 - 1)p_5 = (p_8 - 1)p_8 = (p_9 - 1)p_9 = 0;$$

$$p_4p_5 = p_4p_8 = p_4p_9 = p_5p_8 = p_5p_9 = p_8p_9 = 0;$$

This system have the following 5 solutions:

$$\begin{aligned}
&\textbf{Case 1}: \quad p_4 = 0,\, p_5 = 0,\, p_8 = 1,\, p_9 = 0;\\
&\textbf{Case 2}: \quad p_4 = 0,\, p_5 = 0,\, p_8 = 0,\, p_9 = 1;\\
&\textbf{Case 3}: \quad p_4 = 0,\, p_5 = 1,\, p_8 = 0,\, p_9 = 0;\\
&\textbf{Case 4}: \quad p_4 = 1,\, p_5 = 0,\, p_8 = 0,\, p_9 = 0;\\
&\textbf{Case 5}: \quad p_4 = 0,\, p_5 = 0,\, p_8 = 0,\, p_9 = 0.
\end{aligned}$$

In each case, equating to zero the remaining coefficients in the polynomial $P(u, v, z)$, we obtain a large but rather simple algebraic system for q_i, r_i, s_i, t_i, x_i, y_i. This system contains $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ as parameters.

Solving this system in Case **1**, we obtain systems **1.1- 1.3**; systems **2.1- 2.3** appear in Case **2**; Case **3** produces **3.1- 3.6**; systems **4.1- 4.3** and **5.1- 5.3** correspond Case **4** and Case **5**.

All systems contain four arbitrary parameters $\kappa_1 - \kappa_4$. Notice that additional systems that correspond to particular values of parameters do not exist.

Transformation group

The scalar P_6 system (1) is invariant under the transformations

$$r_1: \qquad \{z, u, v\} \mapsto \{1 - z,\ 1 - u,\ -v\},$$

$$r_2: \qquad \{z, u, v\} \mapsto \{z^{-1},\ z^{-1}u,\ zv\},$$

$$r_3: \qquad \{z, u, v\} \mapsto \{z(z-1)^{-1},\ (z-u)(z-1)^{-1},\ -(z-1)v\}.$$

These involutions change the parameters in the following way

$$r_1: \quad \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\} \mapsto \{\kappa_1,\, 2\kappa_1 - \kappa_2 - \kappa_4,\, \kappa_3,\, \kappa_4\},$$

$$r_2: \quad \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\} \mapsto \{\kappa_1,\, \kappa_4 - 1,\, \kappa_3,\, \kappa_2 + 1\},$$

$$r_3: \quad \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\} \mapsto \{\kappa_1,\, \kappa_2,\, \kappa_3,\, 2\kappa_1 - \kappa_2 - \kappa_4 + 1\}.$$

The involutions r_i and τ act on the set of eighteen non-abelian systems described above. There are three orbits of this action:

$$\textbf{Orbit 1} = \{\mathbf{1.1},\, \mathbf{1.2},\, \mathbf{1.3},\, \mathbf{4.1},\, \mathbf{4.2},\, \mathbf{4.3}\},$$

$$\textbf{Orbit 2} = \{\mathbf{2.1},\, \mathbf{2.2},\, \mathbf{2.3},\, \mathbf{5.1},\, \mathbf{5.2},\, \mathbf{5.3}\},$$

$$\textbf{Orbit 3} = \{\mathbf{3.1},\, \mathbf{3.2},\, \mathbf{3.3},\, \mathbf{3.4},\, \mathbf{3.5},\, \mathbf{3.6}\}.$$

$\mathrm{P}_5 - \mathrm{P}_2$ systems

P_5 systems

In the scalar case, the P_5-system,

$$\begin{cases} zu' &= 2u^3v - 4u^2v - \kappa_1 u^2 + 2uv + (\kappa_1+\kappa_2)u - \kappa_2 + \kappa_4 zu, \\ zv' &= -3u^2v^2 + 4uv^2 - v^2 + 2\kappa_1 uv - (\kappa_1+\kappa_2)v + \kappa_3 - \kappa_4 zv, \end{cases}$$

has the following Okamoto integral:

$$J = u^3v^2 - 2u^2v^2 + uv^2 - \kappa_1 u^2 v + (\kappa_1+\kappa_2)uv - \kappa_3 u - \kappa_2 v + \kappa_3 + \kappa_4 zuv.$$

Note that the system has the structure

$$\begin{cases} zu' &= P_1(u,v) + \kappa_4\, z\, u, \\ zv' &= P_2(u,v) - \kappa_4\, z\, v. \end{cases}$$

We consider the following non-abelian ansatz for the components $P_1(u,v)$ and $P_2(u,v)$:

$$P_1(u,v) = a_1u^3v + a_2u^2vu + a_3uvu^2 + (2-\sum a_i)vu^3 + \\ c_1u^2v + (-4-c_1-c_2)uvu + c_2vu^2 - \kappa_1u^2 + \\ e_1uv + (2-e_1)vu + (\kappa_1+\kappa_2)u - \kappa_2,$$

$$P_2(u,v) = b_1u^2v^2 + b_2uvuv + b_3uv^2u + b_4vu^2v + b_5vuvu + \\ (-3-\sum b_i)v^2u^2 + d_1uv^2 + (4-d_1-d_2)vuv - \\ d_2v^2u - v^2 + f_1uv + (2\kappa_1-f_1)vu - (\kappa_1+\kappa_2)v + \hat{\kappa}_3$$

and it is assumed that the non-Abelian Okamoto integral has the form

$$
\begin{aligned}
J = {} & p_1u^3v^2 + p_2u^2vuv + p_3u^2v^2u + p_4uvu^2v + p_5uvuvu + \\
& p_6uv^2u^2 + p_7vu^3v + p_8vu^2vu + p_9vuvu^2 + (1 - \textstyle\sum p_i)v^2u^3 + \\
& q_1u^2v^2 + q_2uvuv + q_3uv^2u + q_4vu^2v + q_5vuvu + \\
& (-2 - \textstyle\sum q_i)v^2u^2 + r_1u^2v + r_2uvu + (-\kappa_1 - \textstyle\sum r_i)vu^2 + \\
& s_1uv^2 + s_2vuv + (1 - \textstyle\sum s_i)v^2u + t_1uv + \\
& (\kappa_1 + \kappa_2 - t_1)vu - \kappa_3u - \kappa_2v + \hat{\kappa}_3 + z(w_1uv + (\kappa_4 - w_1)vu).
\end{aligned}
$$

There 10 systems of P_5-type that have the Okamoto integral. Under a limiting transition the five classes of P_6 systems turn into five classes of P_5 systems, where each of these classes contain two systems.

P_4 systems

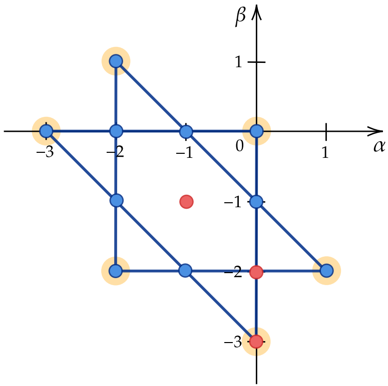
A non-abelian generalization of scalar P_4 system can be written as

$$\begin{cases} u' &= -u^2 + 2uv + \alpha\,[u,v] - 2zu + \kappa_2, \\ v' &= -v^2 + 2vu + \beta\,[v,u] + 2zv + \kappa_3. \end{cases}$$

An ansatz for a non-abelian analog of this system is:

$$\begin{aligned} J = {} & a_1uv^2 + (1 - a_1 - a_2)vuv + a_2v^2u + b_1u^2v + \\ & (-1 - b_1 - b_2)uvu + b_2vu^2 - \kappa_3u + \kappa_4v \\ & +z\,(c_1uv + (-2 - c_1)vu)\,. \end{aligned}$$

All 13 integrable non-Abelian systems of P_4-type found in our previous paper are depicted in the following figure:



Six of these systems have the Okamoto integral. This is one of the three orbits of action of the transformation group. The red point in the middle of the figure is the Hamiltonian non-Abelian system P_4. We denote it by P_4^0.

P_3 systems: $\mathrm{P}_3(D_6)$

In the scalar case, the $\mathrm{P}_3(D_6)$-system,

$$\begin{cases} zu' &= 2u^2v + \kappa_1 u + z\left(\kappa_2 u^2 + \kappa_4\right), \\ zv' &= -2uv^2 - \kappa_1 v + z\left(-2\kappa_2 uv - \kappa_3\right), \end{cases}$$

has the following Okamoto integral:

$$J = u^2v^2 + \kappa_1 uv + z\left(\kappa_2 u^2 v + \kappa_3 u + \kappa_4 v\right).$$

We consider the following non-abelian ansatz for the components $P_1(u,v)$, $P_2(u,v)$, and $Q_2(u,v)$:

$$P_1(u,v) = a_1 u^2 v + (2 - a_1 - a_2)uvu + a_2 vu^2 + \kappa_1 u,$$

$$P_2(u,v) = b_1 uv^2 - (2 + b_1 + b_2)vuv + b_2 v^2 u - \kappa_1 v,$$

$$Q_2(u,v) = c_1 uv + (-2\kappa_2 - c_1)vu - \kappa_3$$

and non-abelian Okamoto integrals of the form

$$J = d_1 u^2 v^2 + d_2 u v^2 u + d_3 uvuv + d_4 v u^2 v + d_5 vuvu$$
$$+ (1 - \sum d_i) v^2 u^2 + e_1 uv + (\kappa_1 - e_1) vu$$
$$+ z \Big(h_1 u^2 v + (\kappa_2 - h_1 - h_2) uvu + h_2 v u^2 + \kappa_3 u + \kappa_4 v \Big).$$

We have 8 polynomial systems of P_3 type. One of them is given by

$$P_1(u,v) = 2uvu + \kappa_1 u, \qquad Q_1(u,u) = \kappa_2 u^2 + \kappa_4,$$
$$P_2(u,v) = -2vuv - \kappa_1 v, \qquad Q_2(u,u) = -2\kappa_2 vu - \kappa_3.$$

The corresponding Okamoto integral is given by

$$J = v u^2 v + \kappa_1 vu + \kappa_3 \, \kappa_2^{-1} [u,v] + z \left(\kappa_2 v u^2 + \kappa_3 u + \kappa_4 v \right).$$

P_2 systems

The scalar P_2-system,

$$\begin{cases} u' &= -u^2 + v - \frac{1}{2}z, \\ v' &= 2uv + \kappa_3, \end{cases}$$

has the following Okamoto integral:

$$J = \frac{1}{2}v^2 - u^2 v - \kappa_3 u - \frac{1}{2}zv.$$

A non-abelian generalization can be written as

$$\begin{cases} u' &= -u^2 + v - \frac{1}{2}z, \\ v' &= 2vu + \beta[v, u] + \kappa_3. \end{cases}$$

In the paper Adler-Sokolov it was proved that $\beta = -1, 0, 1, 2, 3..$

An ansatz for a non-abelian analog is:

$$J = a_1 u^2 v + (-1 - a_1 - a_2) uvu + a_2 vu^2 + \frac{1}{2} v^2 - \kappa_3 u - \frac{1}{2} zv.$$

There two P_2 systems:

$$\left\{ \begin{array}{rcl} u' & = & -u^2 + v - \frac{1}{2} z, \\ v' & = & 2uv + \kappa_3, \end{array} \right.$$

$$J = -u^2 v + \frac{1}{2} v^2 - \kappa_3 u - \frac{1}{2} zv.$$

$$\left\{ \begin{array}{rcl} u' & = & -u^2 + v - \frac{1}{2} z, \\ v' & = & 2vu + \kappa_3, \end{array} \right.$$

$$J = -vu^2 + \frac{1}{2} v^2 - \kappa_3 u - \frac{1}{2} zv.$$

This is $P_2^{(1)}$ system found in Adler-Sokolov.

It is well-known that the scalar system (1) has the isomonodromic representation (7), where matrices $\mathcal{A}(z,\lambda)$ and $\mathcal{B}(z,\lambda)$ have the form

$$\mathcal{A}(z,\lambda) = \frac{A_0}{\lambda} + \frac{A_1}{\lambda-1} + \frac{A_z}{\lambda-z}, \qquad \mathcal{B}(z,\lambda) = -\frac{A_z}{\lambda-z} + B$$

with the following matrices A_0, A_1, A_z, and B:

$$A_0 = \begin{pmatrix} \kappa_4-\kappa_1-1 & uz^{-1}-1 \\ 0 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -uv+\kappa_1 & 1 \\ -u^2v^2+\kappa_1 uv+\kappa_3 & uv \end{pmatrix}$$

$$A_z = \begin{pmatrix} uv+(\kappa_1-\kappa_2-\kappa_4) & -uz^{-1} \\ zuv^2+(\kappa_1-\kappa_2-\kappa_4)zv & -uv \end{pmatrix},$$

$$B = \begin{pmatrix} (z(z-1))^{-1}\left(2u^2v-2zuv-\kappa_1 u-(\kappa_1-\kappa_2-\kappa_4)z\right) & 0 \\ -uv^2-(\kappa_1-\kappa_2-\kappa_4)v & 0 \end{pmatrix}.$$

Degenerations

In the scalar case the following degeneration scheme is well-known:

$$\begin{tikzcd}
 & & \mathrm{P}_3 \arrow[dr] & \\
\mathrm{P}_6 \arrow[r] & \mathrm{P}_5 \arrow[ur] \arrow[dr] & & \mathrm{P}_2 \\
 & & \mathrm{P}_4 \arrow[ur] &
\end{tikzcd}$$

Рис.: Caption

$\mathrm{P}_6 \to \mathrm{P}_5$

After the transformation with the small parameter ε

$$z \mapsto -\varepsilon^{-1} + \varepsilon^{-1} z, \quad \kappa_2 \mapsto \kappa_2 + \varepsilon^{-1} \kappa_4, \quad \kappa_4 \mapsto -\varepsilon \kappa_1 + \varepsilon\, \kappa_4,$$

the Painlevé-6 system (1) becomes the Painlevé-5 system of the form

$$\begin{cases} zu' = 2u^3v - 4u^2v - \kappa_1 u^2 + 2uv + (\kappa_1+\kappa_2)u - \kappa_2 + \kappa_4 zu, \\ zv' = -3u^2v^2 + 4uv^2 - v^2 + 2\kappa_1 uv - (\kappa_1+\kappa_2)v + \kappa_3 - \kappa_4 zv. \end{cases}$$

The corresponding Hamiltonian is

$$zH = u^3v^2 - 2u^2v^2 + uv^2 - \kappa_1 u^2 v + (\kappa_1+\kappa_2)uv - \kappa_3 u - \kappa_2 v + \kappa_3 + \kappa_4 zuv.$$

Supplementing the transformation by the following change of the parameter λ

$$\lambda \mapsto \varepsilon^{-1} z^{-1} (\lambda - 1),$$

one can obtain the Lax pair for the Painlevé-5 system :

$$\mathcal{A}(\lambda, z) = \frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + A_\infty, \qquad \mathcal{B}(\lambda, z) = B_1\lambda + B_\infty,$$

with

$$A_0 = \begin{pmatrix} -uv + \kappa_1 & 1 \\ -u^2v^2 + \kappa_1 uv + \kappa_3 & uv \end{pmatrix}, \qquad A_1 = \begin{pmatrix} uv - \kappa_2 & -u \\ uv^2 - \kappa_2 v & -uv \end{pmatrix},$$

$$A_\infty = \begin{pmatrix} \kappa_4 z & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} \kappa_4 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B_\infty = z^{-1}\begin{pmatrix} 2u^2v - 2uv - \kappa_1 u + \kappa_1 & -u + 1 \\ -u^2v^2 + uv^2 + \kappa_1 uv - \kappa_2 v + \kappa_3 & 0 \end{pmatrix}.$$

$\mathrm{P}_5 \to \mathrm{P}_4$

The Painlevé-5 system after the transformation with the small parameter ε

$$z \mapsto \frac{1}{\sqrt{2}}\varepsilon^{-1}(z-1), \qquad u \mapsto \sqrt{2}\,\varepsilon^{-1}u, \qquad v \mapsto \sqrt{2}\,\varepsilon\, v,$$

$$\kappa_1 = \varepsilon^{-2}, \qquad \kappa_2 \mapsto -2\kappa_2, \qquad \kappa_3 \mapsto 2\varepsilon^2\kappa_3, \qquad \kappa_4 = -2\varepsilon^{-2},$$

becomes the Painlevé-4 system of the form

$$\begin{cases} u' &= -u^2 + 2uv - 2zu + \kappa_2, \\ v' &= -v^2 + 2uv + 2zv + \kappa_3. \end{cases}$$

The corresponding Hamiltonian is

$$H = uv^2 - u^2v - 2zuv - \kappa_3 u + \kappa_2 v.$$

To get the degeneracy of the Lax pair, we consider the following transformation

$$\lambda \mapsto \frac{1}{\sqrt{2}}\,\varepsilon^{-1}(\lambda - 1), \quad \mathcal{A} \mapsto g\mathcal{A}g^{-1}, \quad \mathcal{B} \mapsto g\mathcal{B}g^{-1}, \quad g = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2}\,\varepsilon \end{pmatrix},$$

which brings the pair for Painlevé-5 to the followig pair for Painlevé-4:

$$\mathcal{A}(\lambda, z) = A_1\lambda + A_0 + A_{-1}\lambda^{-1}, \qquad \mathcal{B}(\lambda, z) = B_1\lambda + B_0,$$

where

$$A_1 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} -2z & 1 \\ uv + \kappa_3 & 0 \end{pmatrix},$$

$$A_{-1} = \frac{1}{2}\begin{pmatrix} uv + \kappa_2 & -u \\ uv^2 + \kappa_2 v & -uv \end{pmatrix}, \qquad B_1 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} -u - 2z & 1 \\ uv + \kappa_3 & 0 \end{pmatrix}.$$

$\mathrm{P}_5 \to \mathrm{P}_3$

Under the map

$$z \mapsto z^{\frac{1}{2}}, \qquad u \mapsto \varepsilon^{-1} z^{-\frac{1}{2}}(u-1), \qquad v \mapsto 2\varepsilon\, z^{\frac{1}{2}} v,$$

$$\kappa_1 \mapsto -1 - 2\kappa_1 + 2\kappa_2, \qquad \kappa_2 \mapsto -2\varepsilon\, \kappa_2,$$

$$\kappa_3 \mapsto -4\varepsilon\, \kappa_3, \qquad \kappa_4 \mapsto 2\varepsilon^{-1} \kappa_4,$$

the Painlevé-5 system reduces to the Painlevé-3(D_6)-system of the form

$$\begin{cases} zu' &=& 2u^2 v + \kappa_1 u + z\left(\kappa_2 u^2 + \kappa_4\right), \\ zv' &=& -2uv^2 - \kappa_1 v + z\left(-2\kappa_2 uv - \kappa_3\right). \end{cases}$$

The corresponding Hamiltonian is

$$zH = u^2 v^2 + \kappa_1 uv + z\left(\kappa_2 u^2 v + \kappa_3 u + \kappa_4 v\right).$$

Supplementing the map by the following transformation

$$\lambda \mapsto \frac{1}{2}\varepsilon\, \lambda, \quad \mathcal{A} \mapsto g\mathcal{A}g^{-1}, \quad \mathcal{B} \mapsto g\mathcal{B}g^{-1} + g' g^{-1}, \quad g = \begin{pmatrix} 1 & 0 \\ 0 & 2\varepsilon z^{\frac{1}{2}} \end{pmatrix}$$

and then changing the spectral parameter λ

$$\lambda \mapsto z\lambda$$

to get the Jimbo-Miwa pair, we obtain

$$\mathcal{A}(\lambda, z) = A_0 + A_{-1}\lambda^{-1} + A_{-2}\lambda^{-2}, \quad \mathcal{B}(\lambda, z) = B_1\lambda + B_0 + B_{-1}\lambda^{-1},$$

with

$$A_0 = \begin{pmatrix} \kappa_4 z & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_{-2} = \frac{1}{4}\begin{pmatrix} v + \kappa_2 z & -1 \\ v^2 + \kappa_2 zv & -v \end{pmatrix},$$

$$A_{-1} = \frac{1}{2}\begin{pmatrix} -1 - \kappa_1 & -u \\ -(uv^2 + \kappa_2 zuv + (1 + \kappa_1)v + \kappa_3 z) & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \kappa_4 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_0 = \frac{1}{2}z^{-1}\left(\begin{matrix} 4uv + 2\kappa_2 zu + (1 + \kappa_1) \\ -(uv^2 + \kappa_2 zuv + (1 + \kappa_1)v + \kappa_3 z) \end{matrix}\right.$$

$$B_{-1} = -\frac{1}{4}z^{-1}\begin{pmatrix} v + \kappa_2 z & -1 \\ v^2 + \kappa_2 zv & -v \end{pmatrix} = -z^{-1}A_{-2}.$$

$P_4 \to P_2$

The Painlevé-4 system after the transformation with the small parameter ε

$$z \mapsto \frac{1}{4}\varepsilon^{-4} - \varepsilon^{-1} z, \qquad u \mapsto -\frac{1}{4}\varepsilon^{-2} - \varepsilon\, u, \qquad v \mapsto -\frac{1}{2}\varepsilon^{-1} v,$$

$$\kappa_2 = -\frac{1}{16}\varepsilon^{-6}, \qquad \kappa_3 \mapsto \frac{1}{2}\kappa_3,$$

becomes the Painlevé-2 system of the form

$$\begin{cases} u' &= -u^2 + v - \frac{1}{2} z, \\ v' &= 2uv + \kappa_3, \end{cases}$$

The corresponding Hamiltonian is

$$H = \frac{1}{2}v^2 - u^2 v - \kappa_3 u - \frac{1}{2} z v.$$

The following degeneration data for Painlevé-4 pair

$$\lambda \mapsto \frac{1}{4}\varepsilon^{-2} + 2\varepsilon\lambda, \qquad \mathcal{A} \mapsto g\mathcal{A}g^{-1}, \qquad \mathcal{B} \mapsto g\mathcal{B}g^{-1} + g' g^{-1},$$

where

$$g = \begin{pmatrix} 1 & 0 \\ -\varepsilon\, v & \varepsilon \end{pmatrix},$$

leads to the Jimbo-Miwa pair for the Painlevé-2 system:

$$\mathcal{A}(\lambda, z) = A_2\lambda^2 + A_1\lambda + A_0, \qquad \mathcal{B}(\lambda, z) = B_1\lambda + B_0,$$

where

$$A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & -2 \\ -v & 0 \end{pmatrix},$$

$$A_0 = \begin{pmatrix} -v + z & -2u \\ uv + \kappa_3 & v \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} -u & -1 \\ -\frac{1}{2}v & 0 \end{pmatrix}.$$

$P_3 \to P_2$

The P_3-system under the transformation

$$z \mapsto -2^{\frac{2}{3}}(\varepsilon^{-2} + \varepsilon z), \qquad u \mapsto 2^{-\frac{2}{3}}\varepsilon^{-1}\left(u - 1\right), \qquad v \mapsto 2^{-\frac{1}{3}}\varepsilon v,$$
$$\kappa_1 = 2\varepsilon^{-3}, \qquad \kappa_3 \mapsto 2\kappa_3, \qquad \kappa_4 = \kappa_2 = 1,$$

becomes the Painlevé-2 system.

Degenerations in non-abeian case

All 10 Painlevé-5 systems can be obtained as the result of limiting transitions from Painlevé-6 systems. In addition, two more systems of Painlevé-5 type with $k_4 = 0$ appear as limits.

Let us take three orbit reprentatives of Painlevé-6 and find the whole degeneration scheme for them.

$$
\begin{tikzcd}
 & & {}_3\mathrm{P}_3 \arrow[dr] & \\
{}_6\mathrm{P}_6 \arrow[r] & {}_3\mathrm{P}_5 \arrow[ur] \arrow[dr] & & {}_2\mathrm{P}_2 \\
 & & {}_4\mathrm{P}_4 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \mathrm{P}_4^0 \arrow[ddr] & \\
{}_3\mathrm{P}_6 \arrow[r] & {}_2\mathrm{P}_5 \arrow[ur] \arrow[dr] & & \\
 & & {}_4\mathrm{P}_3 \arrow[r] & \mathrm{P}_2^0 \\
{}_9\mathrm{P}_6 \arrow[r] & {}_5\mathrm{P}_5 \arrow[ur] \arrow[dr] & & \\
 & & {}_4\mathrm{P}_4 \arrow[uur] &
\end{tikzcd}
$$

Рис.: Caption

Here we see another unexpected effect in the degeneration procedure. The systems P_4^0 and P_2^0 have no Okamoto integral. They are well-known Hamiltonian non-abelian Painlevésystems.

Example. Initial system of Painlevé-4 type :

$$\begin{cases} u' &= -u^2 + 2uv - 2zu + \kappa_2, \\ v' &= -v^2 + vu + uv + 2zv + \kappa_3. \end{cases}$$

Initial integral:

$$J = uv^2 - uvu - \kappa_3 u + \kappa_2 v - 2zuv.$$

Step-by-step transformation of the system and integral:

1. Change of variables:

$$z = \frac{1}{4}\varepsilon^{-3} - \varepsilon Z, \quad u(z) = -\frac{1}{4}\varepsilon^{-3} - \varepsilon^{-1}U(Z), \quad v(z) = -2\varepsilon V(Z).$$

System:

$$\begin{cases} U' &= -U^2 + V - \frac{1}{2}Z + \varepsilon^2\left(4UV - 2ZU + \kappa_2\right) + \frac{1}{16}\varepsilon^{-4}, \\ V' &= VU + UV + \frac{1}{2}\kappa_3 + \varepsilon^2\left(-2V^2 + 2ZV\right). \end{cases}$$

Integral:

$$J = \varepsilon(-4UV^2 + 4ZUV - 2\kappa_2 V) + \varepsilon^{-1}(2UVU - V^2 + \kappa_3 U + ZV) \\ + \frac{1}{8}\varepsilon^{-5}V + \varepsilon^{-3}(\frac{1}{2}VU - \frac{1}{2}UV - \frac{1}{4}\kappa_3).$$

2. Change of parameters:

$$\kappa_2 = -\frac{1}{16}\varepsilon^{-6}, \qquad \kappa_3 = 2\kappa.$$

System:

$$\begin{cases} U' &= -U^2 + V - \frac{1}{2}Z + \varepsilon^2(4UV - 2ZU), \\ V' &= VU + UV + \kappa + \varepsilon^2(-2V^2 + 2ZV). \end{cases}$$

Integral:

$$\hat{J} = 4\varepsilon(-UV^2 + ZUV) + \varepsilon^{-1}(2UVU - V^2 + 2\kappa U + ZV) \\ + \tfrac{1}{2}\varepsilon^{-3}(VU - UV - \kappa).$$

3. System after the limit $\varepsilon \to 0$:

$$\begin{cases} U' &= -U^2 + V - \frac{1}{2}Z, \\ V' &= VU + UV + \kappa. \end{cases}$$

Integral after the limit $\varepsilon \to 0$:

$$J_2 = UV - VU.$$