On discrete Hirota-type equations in 3D

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E. V. Ferapontov, V. Novikov and I. Roustemoglou, On the classification of discrete Hirota-type equations in 3D, arXiv:1312.1574, to appear in IMRN (2014).

Notation

Let $u(x^1, x^2, x^3)$ be a function of three independent variables.

Partial derivatives:

$$u_i = u_{x^i}, \ u_{ij} = u_{x^i x^j}, \ \partial_i = \partial_{x^i}, \ etc.$$

Forward/backward ϵ-shifts:

$$T_i, \ T_{\bar{i}}, \qquad e.g. \ T_1 u(x^1, x^2, x^3) = u(x^1 + \epsilon, x^2, x^3).$$

Forward/backward discrete derivatives:

$$\triangle_i = \frac{T_i - 1}{\epsilon}, \quad \triangle_{\bar{i}} = \frac{1 - T_{\bar{i}}}{\epsilon}.$$

Multiple shifts/derivatives:

$$T_{ij} = T_i T_j, \quad \triangle_{i\bar{j}} = \triangle_i \triangle_{\bar{j}}, \quad etc.$$

Motivation

3D Hirota equation (1981):

$$\alpha T_1 v T_{\bar{1}} v + \beta T_2 v T_{\bar{2}} v + \gamma T_3 v T_{\bar{3}} v = 0.$$

Dividing by v^2 and setting $v = e^{u/\epsilon^2}$ one gets

$$\alpha e^{\triangle_{1\bar{1}} u} + \beta e^{\triangle_{2\bar{2}} u} + \gamma e^{\triangle_{3\bar{3}} u} = 0.$$

The corresponding dispersionless limit, $\epsilon \to 0$, is

$$\alpha e^{u_{11}} + \beta e^{u_{22}} + \gamma e^{u_{33}} = 0.$$

Similar examples: Hirota-Miwa, lattice KP, Schwarzian KP, etc.

Motivation

Many 3D discrete integrable equations can be obtained as 'naive' discretisations of dispersionless PDEs, by simply replacing partial derivatives ∂ by discrete derivatives $\triangle$:

Example 1. The PDE

$$(u_1 - u_2)u_{12} + (u_3 - u_1)u_{13} + (u_2 - u_3)u_{23} = 0$$

results in the lattice KP equation,

$$(\triangle_1 u - \triangle_2 u)\triangle_{12} u + (\triangle_3 u - \triangle_1 u)\triangle_{13} u + (\triangle_2 u - \triangle_3 u)\triangle_{23} u = 0.$$

Example 2. The PDE

$$\partial_1 \ln(u_3/u_2) + \partial_2 \ln(u_1/u_3) + \partial_3 \ln(u_2/u_1) = 0$$

results in the Schwarzian KP equation,

$$\triangle_1 \ln(\triangle_3 u/\triangle_2 u) + \triangle_2 \ln(\triangle_1 u/\triangle_3 u) + \triangle_3 \ln(\triangle_2 u/\triangle_1 u) = 0.$$

Method of dispersive deformations

Consider a discrete wave-type equation,

$$\triangle_{t\bar{t}}u - \triangle_{x\bar{x}}f(u) - \triangle_{y\bar{y}}g(u) = 0,$$

equivalently,

$$u_{tt} - f(u)_{xx} - g(u)_{yy} + \frac{\epsilon^2}{12}(u_{tttt} - f(u)_{xxxx} - g(u)_{yyyy}) + \dots = 0.$$

The corresponding dispersionless limit is

$$u_{tt} - f(u)_{xx} - g(u)_{yy} = 0.$$

This PDE possesses solutions of the form $u = R(x, y, t)$ where

$$R_y = \mu(R)R_x, \quad R_t = \lambda(R)R_x,$$

(*one-phase reductions, planar simple waves*): here $\lambda^2 = f' + g'\mu^2$ is the dispersion relation.

Method of dispersive deformations

Let us require that all one-phase reductions of the dispersionless PDE are 'inherited' by the discrete equation:

$$R_y = \mu(R)R_x + \epsilon(a_1 R_{xx} + a_2 R_x^2) + \epsilon^2(a_3 R_{xxx} + a_4 R_x R_{xx} + a_5 R_x^3) + O(\epsilon^3),$$

$$R_t = \lambda(R)R_x + \epsilon(b_1 R_{xx} + b_2 R_x^2) + \epsilon^2(b_3 R_{xxx} + b_4 R_x R_{xx} + b_5 R_x^3) + O(\epsilon^3).$$

This requirement allows us to reconstruct the coefficients $a_i(R), b_i(R)$ in terms of λ, μ. It also leads to strong constraints on $f(u), g(u)$:

$$f'' + g'' = 0, \quad g''(1+f') - g'f'' = 0, \quad f''^2(1+2f') - f'(f'+1)f''' = 0.$$

Setting

$$f(u) = u - \ln(e^u+1), \; g(u) = \ln(e^u+1),$$

we obtain the discrete equation

$$\triangle_{t\bar{t}}u - \triangle_{x\bar{x}}[u - \ln(e^u+1)] - \triangle_{y\bar{y}}[\ln(e^u+1)] = 0,$$

known as the 'gauge-invariant form' of Hirota equation.

Classification results

Discrete quasilinear equations

$$\sum_{i,j=1}^{3} f_{ij}(\triangle u)\triangle_{ij} u = 0,$$

where f_{ij} are functions of $\triangle_1 u, \triangle_2 u, \triangle_3 u$. Dispersionless limit:

$$\sum f_{ij}(u_1, u_2, u_3)u_{ij} = 0.$$

Discrete conservation laws

$$\triangle_1 f + \triangle_2 g + \triangle_3 h = 0$$

where f, g, h are functions of $\triangle_1 u, \triangle_2 u, \triangle_3 u$. Dispersionless limit:

$$\partial_1 f(u_1, u_2, u_3) + \partial_2 g(u_1, u_2, u_3) + \partial_3 h(u_1, u_2, u_3) = 0.$$

Classification results: quasilinear equations

There exists a unique non-degenerate discrete second-order quasilinear equation in 3D, known as the lattice KP equation:

$$(\triangle_1 u - \triangle_2 u)\triangle_{12} u + (\triangle_3 u - \triangle_1 u)\triangle_{13} u + (\triangle_2 u - \triangle_3 u)\triangle_{23} u = 0.$$

Differential-Difference degenerations:
Consider semi-discrete equations of the form

$$f_{11}\triangle_{11} u + f_{12}\triangle_{12} u + f_{22}\triangle_{22} u + f_{13}\triangle_1 u_3 + f_{23}\triangle_2 u_3 + f_{33} u_{33} = 0,$$

where $f_{ij} = f_{ij}(\triangle_1 u, \triangle_2 u, u_3)$. There exists a unique non-degenerate semi-discrete equation of this type,

$$(\triangle_1 u - \triangle_2 u)\triangle_{12} u - \triangle_1 u_3 + \triangle_2 u_3 = 0.$$

known as the semi-discrete Toda lattice.

Classification results: conservative equations

Integrable discrete conservation laws,

$$\triangle_1 f + \triangle_2 g + \triangle_3 h = 0,$$

are grouped into seven three-parameter families,

$$\alpha I + \beta J + \gamma K = 0,$$

where $\alpha, \beta, \gamma = const$, and I, J, K are three linearly independent discrete conservation laws of seven octahedron-type equations of the form

$$F(T_1 u, T_2 u, T_3 u, T_{12} u, T_{13} u, T_{23} u) = 0.$$

Example 1: lattice KP equation

$$(T_1u - T_2u)T_{12}u + (T_3u - T_1u)T_{13}u + (T_2u - T_3u)T_{23}u = 0.$$

Three conservation laws:

$$I = \triangle_1[(\triangle_3 u)^2 - (\triangle_2 u)^2] + \triangle_2[(\triangle_1 u)^2 - (\triangle_3 u)^2] + \triangle_3[(\triangle_2 u)^2 - (\triangle_1 u)^2] = 0,$$

$$J = \triangle_1 \ln(\triangle_3 u - \triangle_2 u) - \triangle_2 \ln(\triangle_1 u - \triangle_3 u) = 0,$$

$$K = \triangle_1 \ln(\triangle_3 u - \triangle_2 u) - \triangle_3 \ln(\triangle_2 u - \triangle_1 u) = 0.$$

Example 2: Schwarzian KP equation

$$(T_2\triangle_1 u)(T_3\triangle_2 u)(T_1\triangle_3 u) = (T_2\triangle_3 u)(T_3\triangle_1 u)(T_1\triangle_2 u).$$

Three conservation laws:

$$I = \triangle_2 \ln\left(1 - \frac{\triangle_3 u}{\triangle_1 u}\right) - \triangle_3 \ln\left(\frac{\triangle_2 u}{\triangle_1 u} - 1\right) = 0,$$

$$J = \triangle_3 \ln\left(1 - \frac{\triangle_1 u}{\triangle_2 u}\right) - \triangle_1 \ln\left(\frac{\triangle_3 u}{\triangle_2 u} - 1\right) = 0,$$

$$K = \triangle_1 \ln\left(1 - \frac{\triangle_2 u}{\triangle_3 u}\right) - \triangle_2 \ln\left(\frac{\triangle_1 u}{\triangle_3 u} - 1\right) = 0.$$

Classification results: semi-discrete equations

Integrable semi-discrete conservation laws,

$$\triangle_1 f + \triangle_2 g + \partial_3 h = 0,$$

are grouped into seven three-parameter families,

$$\alpha I + \beta J + \gamma K = 0,$$

where $\alpha, \beta, \gamma = const$, and I, J, K are three linearly independent semi-discrete conservation laws of seven differential-difference equations of the form

$$F(u, T_1 u, T_2 u, u_3, T_{12} u, T_1 u_3, T_2 u_3) = 0.$$

Example: Adler's equation

$$T_{12}v = \frac{T_1vT_2v}{v} + \frac{T_2vT_1v_3 - T_1vT_2v_3}{T_2v - T_1v}.$$

Conservation laws (set $v = e^{u/\epsilon}, \partial_3 \to \frac{1}{\epsilon}\partial_3$):

$$I = \triangle_1(e^{\triangle_2 u} - u_3) + \partial_3 \ln\left(e^{\triangle_1 u} - e^{\triangle_2 u}\right) = 0,$$

$$J = \triangle_2(e^{\triangle_1 u} - u_3) + \partial_3 \ln\left(e^{\triangle_1 u} - e^{\triangle_2 u}\right) = 0,$$

$$K = \triangle_1(e^{2\triangle_2 u} - 2e^{\triangle_2 u}u_3 + u_3^2) + \triangle_2(2e^{\triangle_1 u}u_3 - e^{2\triangle_1 u} - u_3^2) + \partial_3(2e^{\triangle_2 u} - 2e^{\triangle_1 u}) = 0.$$

List of 7 differential-difference equations

$$\frac{T_{12}v}{T_2v}+\frac{T_1v_3}{T_1v}=\frac{T_1v}{v}+\frac{T_2v_3}{T_2v},$$
$$T_{12}v=\frac{T_1vT_2v}{v}+\frac{T_2vT_1v_3-T_1vT_2v_3}{T_2v-T_1v},$$
$$\frac{vT_{12}v}{T_1v}=\frac{T_1vT_2v_3}{T_1v_3},$$
$$(T_{12}u-T_2u)T_1u_3=(T_1u-u)T_2u_3,$$
$$v(T_{12}v-T_2v)T_1v_3=T_1v(T_1v-v)T_2v_3,$$
$$(T_2\triangle_1u)(\triangle_2u)T_1u_3=(T_1\triangle_2u)(\triangle_1u)T_2u_3,$$
$$(T_2\sinh\triangle_1u)(\sinh\triangle_2u)T_1u_3=(T_1\sinh\triangle_2u)(\sinh\triangle_1u)T_2u_3.$$

Here $v=e^{u/\epsilon}$. Lax pairs are computed for all 7 cases. Their structure is

$$T_2\psi=p(u,T_1u,T_2u)T_1\psi+q(u,T_1u,T_2u)\psi,$$
$$\epsilon\psi_3=r(u,u_3,T_1u,T_2u)T_1\psi+s(u,u_3,T_1u,T_2u)\psi.$$

Sketch of classification

The corresponding dispersionless limit is

$$\partial_1 f + \partial_2 g + \partial_3 h = 0$$

where f, g, h are functions of $u_1,\ u_2,\ u_3$. Applying our deformation method, at the order ϵ we obtain

$$f_{u_1} = g_{u_2} = h_{u_3} = 0, \quad f_{u_2} + g_{u_1} + f_{u_3} + h_{u_1} + g_{u_3} + h_{u_2} = 0,$$

Up to a non-zero factor, any integrable equation of this type is equivalent to

$$[p(u_1) - q(u_2)]u_{12} + [r(u_3) - p(u_1)]u_{13} + [q(u_2) - r(u_3)]u_{23} = 0,$$

where the functions $p(u_1), q(u_2), r(u_3)$ satisfy some second-order integrability conditions.

...Sketch of classification

Step 1. First, we solve the integrability conditions. Modulo unessential translations and rescalings this leads to seven quasilinear integrable PDEs.

Step 2. For all of these seven PDEs, we calculate first-order conservation laws. Any integrable second-order quasilinear PDE possesses exactly four conservation laws.

Step 3. Taking linear combinations of the four conservation laws, and replacing ∂ by $\triangle$, we obtain discrete equations which are the *candidates* for integrability.

Step 4. Applying the ϵ^2-integrability test, we find that only linear combinations of three (out of four) conservation laws pass the integrability test. Moreover, each triplet of conservation laws corresponds to one and the same octahedron-type equation.

Numerics

Here we use Mathematica to compare numerical solutions for the discrete equation,

$$\triangle_{t\bar{t}}\ u - \triangle_{x\bar{x}}\ [u - \ln(e^u + 1)] - \triangle_{y\bar{y}}\ [\ln(e^u + 1)] = 0,$$

and its dispersionless limit,

$$u_{tt} - [u - \ln(e^u + 1)]_{xx} - [\ln(e^u + 1)]_{yy} = 0,$$

with the following Cauchy data:

Disrete equation: $u(x, y, 0) = 3e^{-(x^2+y^2)}$, $u(x, y, -\epsilon) = 3e^{-(x^2+y^2)}$.

Dispersionless equation: $u(x, y, 0) = 3e^{-(x^2+y^2)}$, $u_t(x, y, 0) = 0$.

Numerics: dispersionless equation

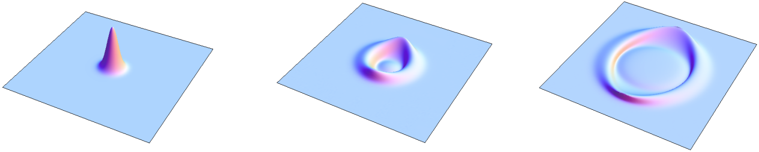

Figure: Numerical solution of the dispersionless equation for $t = 0, 4, 8$.

According to the general theory this solution is expected to break down in finite time.

Numerics: discrete equation

The discrete equation can be equivalently written as

$$u(t+\epsilon) = -u(t-\epsilon) + (T_x + T_{\bar{x}})(u - \ln(e^u + 1)) + (T_y + T_{\bar{y}})\ln(e^u + 1),$$

with

$$u(x,y,0) = 3e^{-(x^2+y^2)},\ u(x,y,-\epsilon) = 3e^{-(x^2+y^2)}.$$

No breakdown in this case.

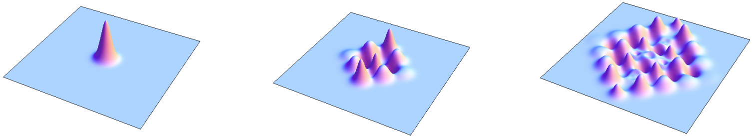

Figure: Solution of the discrete equation for $\epsilon = 2$ and $t = 0, 4, 8$.

Numerics: discrete equation

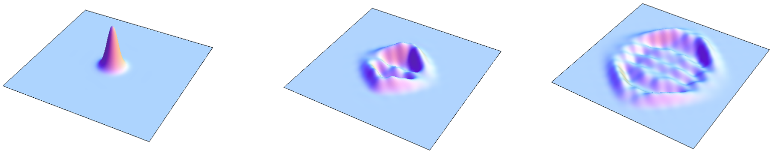

Figure: Solution of the discrete equation for $\epsilon = 1$ and $t = 0, 4, 8$.

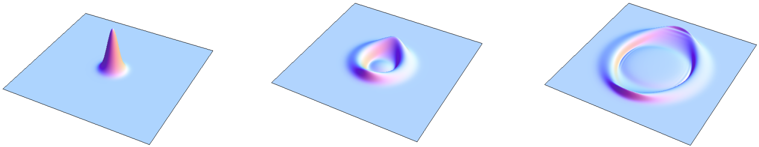

Figure: Numerical solution of the discrete equation for $\epsilon = 1/8$ and $t = 0, 4, 8$.

Numerics: discrete equation

As $\epsilon \to 0$, solution of the discrete equation tends to solution of the dispersionless equation until the breakdown occurs.

At the breaking point:

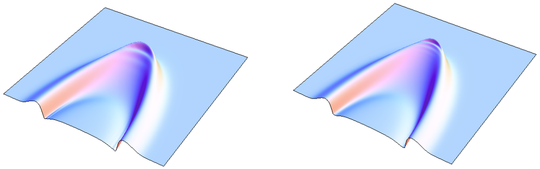

Figure: Formation of a dispersive shock wave in the numerical solution of the discrete equation for $\epsilon = 1/8$ (left) and $\epsilon = 1/16$ (right), at $t = 8$.

Concluding remark

- CKP equation is

$$\begin{gathered}(e^{\epsilon\triangle_{123}u+\triangle_{23}u+\triangle_{13}u+\triangle_{12}u}-e^{\triangle_{23}u}-e^{\triangle_{13}u}-e^{\triangle_{12}u})^2=\\ 4(e^{\triangle_{13}u+\triangle_{23}u}+e^{\triangle_{12}u+\triangle_{13}u}+e^{\triangle_{12}u+\triangle_{23}u}\\ -e^{\epsilon\triangle_{123}u+\triangle_{23}u+\triangle_{13}u+\triangle_{12}u}-e^{\triangle_{23}u+\triangle_{13}u+\triangle_{12}u}).\end{gathered}$$

Its dispersionless limit,

$$(e^{u_{23}+u_{13}+u_{12}}-e^{u_{23}}-e^{u_{13}}-e^{u_{12}})^2=4(e^{u_{13}+u_{23}}+e^{u_{12}+u_{13}}+e^{u_{12}+u_{23}}-2e^{u_{23}+u_{13}+u_{12}}),$$

is equivalent to the product of four copies of the dispersionless BKP equation: setting $u=2v$ we obtain

$$(e^{v_{23}+v_{13}+v_{12}}+e^{v_{23}}+e^{v_{13}}+e^{v_{12}})(e^{v_{23}+v_{13}+v_{12}}-e^{v_{23}}-e^{v_{13}}+e^{v_{12}})\times$$

$$(e^{v_{23}+v_{13}+v_{12}}-e^{v_{23}}+e^{v_{13}}-e^{v_{12}})(e^{v_{23}+v_{13}+v_{12}}+e^{v_{23}}-e^{v_{13}}-e^{v_{12}})=0.$$

Hydrodynamic reductions of each BKP-branch of the dispersionless equation are inherited by the full CKP equation.