Bi-Hamiltonian Structure of WDVV Equations

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19.09.2014

WDVV, N=3

- Associativity WDVV equations

$$\eta^{\mu\lambda}\frac{\partial^3 F}{\partial t^\lambda \partial t^\alpha \partial t^\beta}\frac{\partial^3 F}{\partial t^\nu \partial t^\mu \partial t^\gamma}=\eta^{\mu\lambda}\frac{\partial^3 F}{\partial t^\nu \partial t^\alpha \partial t^\mu}\frac{\partial^3 F}{\partial t^\lambda \partial t^\beta \partial t^\gamma}$$

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- $N = 3$. $\eta_{\alpha\beta} = \delta_{\alpha+\beta,4}$ and $F = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 + f(t^2, t^3)$,

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$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}.$$

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$$f_{ttt}=f_{xxt}^2-f_{xxx}f_{xtt}.$$

- We introduce new variables $a^1=a=f_{xxx}$, $a^2=b=f_{xxt}$, $a^3=c=f_{xtt}$. Then the compatibility conditions for the WDVV equation can be written as an hydrodynamic type system of PDEs

$$a_t^i=v_j^i(\mathbf{a})a_x^j;$$

more precisely, $a_t=b_x$, $b_t=c_x$, $c_t=(b^2-ac)_x$.

WDVV, N=3, Bi-Hamiltonian Structure

- WDVV equations are bi-Hamiltonian, i.e.

$$a^i_t = A_1^{ij}\frac{\delta \mathbf{H}_2}{\delta a^j} = A_2^{ij}\frac{\delta \mathbf{H}_1}{\delta a^j},$$

with respect to two compatible local Hamiltonian operators $\hat{A}_1$ and $\hat{A}_2$, with expressions

$$\hat{A}_1 = \begin{pmatrix} -\frac{3}{2}\partial_x & \frac{1}{2}\partial_x a & \partial_x b \\ \frac{1}{2}a\partial_x & \frac{1}{2}(\partial_x b + b\partial_x) & \frac{3}{2}c\partial_x + c_x \\ b\partial_x & \frac{3}{2}\partial_x c - c_x & (b^2 - ac)\partial_x + \partial_x(b^2 - ac) \end{pmatrix}$$

$$\hat{A}_2 = \begin{pmatrix} 0 & 0 & \partial_x^3 \\ 0 & \partial_x^3 & -\partial_x^2 a\partial_x \\ \partial_x^3 & -\partial_x a\partial_x^2 & \partial_x^2 b\partial_x + \partial_x b\partial_x^2 + \partial_x a\partial_x a\partial_x \end{pmatrix}$$

and Hamiltonian densities, respectively, $h_2 = c$,
$h_1 = -\frac{1}{2}a(\partial_x^{-1}b)^2 - (\partial_x^{-1}b)(\partial_x^{-1}c)$, where $\mathbf{H}_i = \int h_i dx$.

WDVV, N=4, Hydrodynamic Form

- $N = 4$. $\eta_{\alpha\beta} = \delta_{\alpha+\beta,5}$ and $F = \frac{1}{2}(t^1)^2 t^4 + t^1 t^2 t^3 + f(t^2, t^3, t^4)$.

WDVV, N=4, Hydrodynamic Form

- $N=4$. $\eta_{\alpha\beta}=\delta_{\alpha+\beta,5}$ and $F=\frac{1}{2}(t^1)^2t^4+t^1t^2t^3+f(t^2,t^3,t^4)$. (after the identifications $x=t^2$, $y=t^3$, $z=t^4$)

$$-2f_{xyz}-f_{xyy}f_{xxy}+f_{yyy}f_{xxx}=0,$$
$$-f_{xzz}-f_{xyy}f_{xxz}+f_{yyz}f_{xxx}=0,$$
$$-2f_{xyz}f_{xxz}+f_{xzz}f_{xxy}+f_{yzz}f_{xxx}=0,$$
$$-f_{yyy}f_{xxz}+f_{yzz}+f_{yyz}f_{xxy}=0,$$
$$f_{zzz}-(f_{xyz})^2+f_{xzz}f_{xyy}-f_{yyz}f_{xxz}+f_{yzz}f_{xxy}=0,$$
$$f_{yyy}f_{xzz}-2f_{yyz}f_{xyz}+f_{yzz}f_{xyy}=0.$$

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$$
\begin{aligned}
-2f_{xyz}-f_{xyy}f_{xxy}+f_{yyy}f_{xxx}&=0,\\
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f_{yyy}f_{xzz}-2f_{yyz}f_{xyz}+f_{yzz}f_{xyy}&=0.
\end{aligned}
$$

- We introduce new field variables a^k in correspondence with every derivative $f_{t^it^jt^k}$ which contains at least one instance of $x=t^2$, i.e. $a^1=f_{xxx}, a^2=f_{xxy}, a^3=f_{xxz}, a^4=f_{xyy}, a^5=f_{xyz}, a^6=f_{xzz}$.
- The compatibility conditions for this system can be written as a pair of hydrodynamic type systems

$$a^i_y=v^i_j(\mathbf{a})a^j_x,\quad a^i_z=w^i_j(\mathbf{a})a^j_x.$$

WDVV, N=4, Hydrodynamic Form

- More precisely

$$a^i_y = (v^i(\mathbf{a}))_x, \quad a^i_z = (w^i(\mathbf{a}))_x,$$

where

$$v^1 = a^2, \quad w^1 = a^3, \quad v^2 = a^4, \quad v^3 = w^2 = a^5, \quad w^3 = a^6,$$

$$v^4 = f_{yyy} = \frac{2a^5 + a^2 a^4}{a^1}, \quad v^5 = w^4 = f_{yyz} = \frac{a^3 a^4 + a^6}{a^1},$$

$$v^6 = w^5 = f_{yzz} = \frac{2a^3 a^5 - a^2 a^6}{a^1},$$

$$w^6 = f_{zzz} = (a^5)^2 - a^4 a^6 + \frac{(a^3)^2 a^4 + a^3 a^6 - 2a^2 a^3 a^5 + (a^2)^2 a^6}{a^1}.$$

WDVV, N=4, First Hamiltonian Structure

- Under the point transformations

$$a^2 = \frac{1}{2}(u^1 + u^2 + u^3 + u^4),$$
$$a^3 = \frac{1}{4}[(u^1)^2+(u^2)^2+(u^3)^2+(u^4)^2] - \frac{1}{8}(u^1+u^2+u^3+u^4)^2 - \frac{1}{2}a^1a^4,$$
$$a^5 = \frac{1}{2a^1}(2a^2a^3 + u^1u^2u^3 + u^1u^2u^4 + u^1u^3u^4 + u^2u^3u^4),$$
$$a^6 = \frac{1}{a^1}[(a^3)^2 - u^1u^2u^3u^4].$$

- WDVV equations take the Hamiltonian form

$$u^i_y = K^{ij}\partial_x \frac{\delta \mathbf{H}_7}{\delta u^j}, \quad u^i_z = K^{ij}\partial_x \frac{\delta \mathbf{H}_8}{\delta u^j},$$

where the Hamiltonian densities $h_7 = a^5$ and $h_8 = \frac{1}{2}a^6$, while the momentum density $h_6 = a^3$.

WDVV, N=4, First Hamiltonian Structure

$$K^{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Bi-Hamiltonian Structure

- Bi-Hamiltonian hierarchy

$$u^i_{t^k} = A_2^{is}\frac{\delta \mathbf{H}_k}{\delta u^s} = A_1^{is}\frac{\delta \mathbf{H}_{k+1}}{\delta u^s}, \quad i = 1, ..., N$$

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- Dubrovin–Novikov type *first* order Hamiltonian operator

$$A_1^{ij} = g_1^{ij}(\mathbf{u})\partial_x + b_{1k}^{ij}(\mathbf{u})u_x^k,$$

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- Dubrovin–Novikov type *third* order Hamiltonian operator

$$A_2^{ij} = g_2^{ij}(\mathbf{u})\partial_x^3 + b_{2k}^{ij}(\mathbf{u})u_x^k\partial_x^2 + [c_{2k}^{ij}(\mathbf{u})u_{xx}^k + c_{2km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x$$

$$+d_{2k}^{ij}(\mathbf{u})u_{xxx}^k + d_{2km}^{ij}(\mathbf{u})u_{xx}^k u_x^m + d_{2kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n,$$

while the Hamiltonian functionals $\mathbf{H}_k = \int h_k(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...)dx$.

Local Casimirs

- In local Casimirs $a^k(\mathbf{u})$

$$A_2^{ij} = \partial_x(g^{ij}\partial_x + c_k^{ij}a_x^k)\partial_x.$$

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- The Potemin System

$$c_{nkm} = \frac{1}{3}(g_{nm,k} - g_{nk,m}),$$
$$g_{mk,n} + g_{kn,m} + g_{mn,k} = 0,$$
$$c_{mnk,l} = -g^{pq} c_{pml} c_{qnk}.$$

where $c_{ijk} = g_{iq} g_{jp} c_k^{pq}$.

Bi-Hamiltonian Structure

- **Our main observation** is: If integrable hierarchy has quasi-homogeneous conservation law densities

$$h_1 = a_{sm}(\mathbf{u})u_x^s u_x^m,$$

$$h_2 = a_{ms}^{(1)}(\mathbf{u})u_{xx}^m u_{xx}^s + a_{lms}^{(2)}(\mathbf{u})u_{xx}^l u_x^m u_x^s + a_{lsmn}^{(3)}(\mathbf{u})u_x^l u_x^s u_x^m u_x^n, \ldots,$$

then the first member of this hierarchy

$$u_{t^1}^i = A_2^{is}\frac{\delta \mathbf{H}_1}{\delta u^s} = A_1^{is}\frac{\delta \mathbf{H}_2}{\delta u^s}, \quad i = 1, \ldots, N$$

becomes

$$u_t^i = (g_2^{ip}\partial_x^3 + \text{l.o.t.})(-2a_{pm}u_{xx}^m + \text{l.o.t.})$$
$$= (g_1^{ip}\partial_x + \text{l.o.t.})(2a_{pm}^{(1)}u_{xxxx}^m + \text{l.o.t.}).$$

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- Then

$$g_2^{ip}a_{pm} = -g_1^{ip}a_{pm}^{(1)}.$$

Second Hamiltonian Structure

- Thus (if $\det a_{jm} \neq 0$) then

$$g_2^{ij} = -g_1^{ip} a_{pm}^{(1)} c^{mj},$$

where $a_{im} c^{mj} = \delta_i^j$ and $c^{ip} a_{pj} = \delta_j^i$.

- In flat coordinates u^k we have $g_1^{ij} = K^{ij}$. Thus

$$g^{ij}(\mathbf{u}) = -K^{ip} a_{pm}^{(1)} c^{mj}.$$

Second Hamiltonian Structure

The metric $g_{in}(\mathbf{u})$ is transformed from the coordinates u^k to the coordinates $a^k(\mathbf{u})$ as

$$g_{ik}(\mathbf{a}) = \begin{pmatrix} (a^4)^2 & -2a^5 & 2a^4 & -(a^1a^4+a^3) & a^2 & 1 \\ -2a^5 & -2a^3 & a^2 & 0 & a^1 & 0 \\ 2a^4 & a^2 & 2 & -a^1 & 0 & 0 \\ -(a^1a^4+a^3) & 0 & -a^1 & (a^1)^2 & 0 & 0 \\ a^2 & a^1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The metric $g_{ik}(\mathbf{a})$ is a Monge metric satisfying Potemin system and it generates a Dubrovin–Novikov type third-order Hamiltonian operator $\hat{A}_2$ in canonical form

$$A_2^{ij} = \partial_x(g^{ij}\partial_x + c_k^{ij}a_x^k)\partial_x.$$

Linear System

- The linear system of PDEs

$$\psi_{j,k} = \frac{1}{3}\psi_p g^{pq}(g_{qj,k} - g_{qk,j})$$

on N functions $\psi_k(\mathbf{a})$ can be interpreted as N commuting linear systems of ODEs for each fixed index k. Thus this linear system possesses a general solution parameterised by N arbitrary constants only.

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- It is easy to see that $(\psi_{j,k})_n = (\psi_{j,n})_k = 0$. This means that ψ_k are *linear* functions with respect to field variables a^s, i.e.

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- Let us choose N particular solutions ψ_k^γ of this system ($\det \psi_p^\gamma \neq 0$)

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- Then we can identify $\psi_{j,k}^\gamma = \psi_{jk}^\gamma$.

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- Then we can identify $\psi_{j,k}^\gamma = \psi_{jk}^\gamma$. Here $\psi_{jk}^\gamma = -\psi_{kj}^\gamma$.

Metric Decomposition

The metric coefficients $g_{ik}(\mathbf{a})$ of Potemin system can be presented in the form

$$g_{ij} = \phi_{\beta\gamma}\psi_i^\beta\psi_j^\gamma,$$

where $\phi_{\beta\gamma}$ are constants ($\det\phi_{\beta\gamma} \neq 0$),

$$c_{ijk} = -\phi_{\beta\gamma}\psi_i^\beta\psi_{jk}^\gamma$$

and

$$\phi_{\beta\gamma}(\psi_{is}^\beta\psi_{jk}^\gamma + \psi_{js}^\beta\psi_{ki}^\gamma + \psi_{ks}^\beta\psi_{ij}^\gamma) = 0, \quad \phi_{\beta\gamma}(\omega_i^\beta\psi_{jk}^\gamma + \omega_j^\beta\psi_{ki}^\gamma + \omega_k^\beta\psi_{ij}^\gamma) = 0.$$

Also components of the inverse metric $g^{ik}(\mathbf{a})$ have the form

$$g^{ij} = \phi^{\beta\gamma}\psi_\beta^i\psi_\gamma^j,$$

where $\phi^{\beta\gamma}$ and ψ_β^i are inverse matrices to $\phi_{\beta\gamma}$ and ψ_k^γ, respectively.

Factorisation. Second Hamiltonian Structure

- Substitution factorised expressions $g_{ij} = \phi_{\beta\gamma}\psi_i^\beta\psi_j^\gamma$ and $c_{ijk} = -\phi_{\beta\gamma}\psi_i^\beta\psi_{jk}^\gamma$

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$$a_t^i = \partial_x(g^{is}\partial_x + c_k^{is}a_x^k)\partial_x\frac{\delta\mathbf{H}}{\delta a^s}$$

- leads to

$$a_t^i = \phi^{\beta\gamma}\partial_x\psi_\beta^i\partial_x\psi_\gamma^s\partial_x\frac{\delta\mathbf{H}}{\delta a^s}.$$

- Thus a Dubrovin–Novikov type third order Hamiltonian operator is

$$A_2^{ij} = \phi^{\beta\gamma}\partial_x\psi_\beta^i\partial_x\psi_\gamma^j\partial_x.$$

Nonlocal Casimirs

- Under the potential substitution $a^i = b^i_x$ the evolution system

$$a^i_t = (V^i(\mathbf{a}, \mathbf{a}_x, \mathbf{a}_{xx}, ...))_x$$

takes the form

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Correspondingly, if this evolution system has a Dubrovin–Novikov type *third* order Hamiltonian structure

$$a^i_t = \partial_x(g^{is}\partial_x + c^{is}_k a^k_x)\partial_x \frac{\delta \mathbf{H}}{\delta a^s} = \phi^{\beta\gamma}\partial_x\psi^i_\beta\partial_x\psi^s_\gamma\partial_x\frac{\delta \mathbf{H}}{\delta a^s},$$

then in potential variables b^i we see that evolution system has a local Hamiltonian structure of *first* order

$$b^i_t = -(g^{is}\partial_x + c^{is}_k b^k_{xx})\frac{\delta \mathbf{H}}{\delta b^s} = -\phi^{\beta\gamma}\psi^i_\beta\partial_x\psi^s_\gamma\frac{\delta \mathbf{H}}{\delta b^s},$$

but this is **not** a Dubrovin–Novikov type first order Hamiltonian structure, because its coefficients $g^{is}(\mathbf{b}_x)$, $c^{is}_k(\mathbf{b}_x)$, $\psi^i_\beta(\mathbf{b}_x)$ have no geometrical interpretation.

Nonlocal Casimirs

- Casimirs $\mathbf{S}^\beta = \int s^\beta dx$ generate "zeroth" flows, i.e.

$$0 = (g^{is}\partial_x + c_k^{is} b_{xx}^k)\frac{\delta \mathbf{S}^\alpha}{\delta b^s} = \phi^{\beta\gamma}\psi_\beta^i \partial_x \psi_\gamma^s \frac{\delta \mathbf{S}^\alpha}{\delta b^s}.$$

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Theorem: *Casimirs are determined by*

$$\mathbf{S}^\alpha = \int \left(\frac{1}{2}\psi^\alpha_{mk} b^k_x + \omega^\alpha_m\right) b^m dx.$$

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Theorem: *Casimirs are determined by*

$$\mathbf{S}^\alpha = \int \left(\frac{1}{2}\psi_{mk}^\alpha b_x^k + \omega_m^\alpha\right) b^m dx.$$

Proof: Taking into account $\psi_{sm}^\alpha = -\psi_{ms}^\alpha$ variation derivatives are

$$\frac{\delta \mathbf{S}^\alpha}{\delta b^s} = \psi_s^\alpha.$$

Then "zeroth" flows imply

$$0 = \phi^{\beta\gamma}\psi_\beta^i \partial_x \psi_\gamma^s \frac{\delta \mathbf{S}^\alpha}{\delta b^s} = \phi^{\beta\gamma}\psi_\beta^i \partial_x \psi_\gamma^s \psi_s^\alpha.$$

Taking into account $\psi_\gamma^s \psi_s^\alpha = \delta_\gamma^\alpha$, one can see that really $\phi^{\beta\gamma}\psi_\beta^i \partial_x \delta_\gamma^\alpha = 0$. The Theorem is proved.

- Any Hamiltonian system

$$b^i_t = -(g^{is}\partial_x + c^{is}_k b^k_{xx})\frac{\delta \mathbf{H}}{\delta b^s} = -\phi^{\beta\gamma}\psi^i_\beta \partial_x \psi^s_\gamma \frac{\delta \mathbf{H}}{\delta b^s}$$

has the momentum $\mathbf{P} = \int P dx$, this means that

$$b^i_x = -\phi^{\beta\gamma}\psi^i_\beta \partial_x \psi^s_\gamma \frac{\delta \mathbf{P}}{\delta b^s}.$$

Then the momentum density P can be reconstructed, because all variational derivatives are known:

$$\frac{\delta \mathbf{P}}{\delta b^k} = -\phi_{\beta\gamma}\psi^\beta_k \partial_x^{-1}\psi^\gamma_m b^m_x.$$

Momentum

- Any Hamiltonian system

$$b^i_t = -(g^{is}\partial_x + c^{is}_k b^k_{xx})\frac{\delta \mathbf{H}}{\delta b^s} = -\phi^{\beta\gamma}\psi^i_\beta \partial_x \psi^s_\gamma \frac{\delta \mathbf{H}}{\delta b^s}$$

has the momentum $\mathbf{P} = \int P dx$, this means that

$$b^i_x = -\phi^{\beta\gamma}\psi^i_\beta \partial_x \psi^s_\gamma \frac{\delta \mathbf{P}}{\delta b^s}.$$

Then the momentum density P can be reconstructed, because all variational derivatives are known:

$$\frac{\delta \mathbf{P}}{\delta b^k} = -\phi_{\beta\gamma}\psi^\beta_k \partial_x^{-1} \psi^\gamma_m b^m_x.$$

Theorem: *A momentum of the above Hamiltonian system* is

$$\mathbf{P} = -\int \left(\frac{1}{3}\phi_{\beta\gamma}\omega^\beta_q \psi^\gamma_{pm} b^m_x + \frac{1}{2}\phi_{\beta\gamma}\omega^\beta_p \omega^\gamma_q\right) b^p b^q dx.$$

Hamiltonian

- Hamiltonian system

$$b^i_t = v^i(\mathbf{b}_x) = -\phi^{\beta\gamma}\psi^i_\beta \partial_x \psi^s_\gamma \frac{\delta \mathbf{H}}{\delta b^s}.$$

Thus

$$\frac{\delta \mathbf{H}}{\delta b^k} = -\phi_{\beta\gamma}\psi^\beta_k \partial_x^{-1} \psi^\gamma_m v^m(\mathbf{b}_x).$$

In a general case the expression $\psi^\gamma_m v^m(\mathbf{b}_x) = (\psi^\gamma_{mk} b^k_x + \omega^\gamma_m) v^m(\mathbf{b}_x)$ nonlinearly depends on b^k_x. However its primitive exists if and only if this expression depends on b^k_x linearly, i.e. $\psi^\gamma_m v^m(\mathbf{b}_x) = \eta^\gamma_m b^m_x$, where η^γ_m is a constant matrix.

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- **Theorem**: *Non-diagonalisable Hamiltonian hydrodynamic type system has the Hamiltonian*

$$\mathbf{H} = \frac{1}{2}\int(\zeta_{pqm}b^m_x - \phi_{\beta\gamma}\omega^\beta_p\eta^\gamma_q)b^p b^q dx,$$

where ζ_{pqm} are symmetric constant matrices with respect to first two indices p, q such that

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$$\zeta_{kpq} = \frac{1}{3}\phi_{\beta\gamma}(\psi_{kp}^\beta \eta_q^\gamma + 2\psi_{qk}^\beta \eta_p^\gamma),$$

where the constant matrix η_q^γ must satisfy the set of contraints

$$\phi_{\beta\gamma}(\psi_{qp}^\beta \eta_k^\gamma + \psi_{kq}^\beta \eta_p^\gamma + \psi_{pk}^\beta \eta_q^\gamma) = 0.$$

Conservation Laws

- Any Hamiltonian system

$$b_t^i = -\phi^{\beta\gamma}\psi_\beta^i \partial_x \psi_\gamma^s \frac{\delta \mathbf{H}}{\delta b^s}$$

possesses N conservation laws associated with Casimirs:

$$s_t^\alpha = -\left[\frac{1}{2}\psi_{mk}^\alpha b^m \phi^{\beta\gamma}\psi_\beta^k \left(\psi_\gamma^s \frac{\delta \mathbf{H}}{\delta b^s}\right)_x + \phi^{\alpha\gamma}\psi_\gamma^s \frac{\delta \mathbf{H}}{\delta b^s}\right]_x,$$

where

$$s^\alpha = \left(\frac{1}{2}\psi_{mk}^\alpha b_x^k + \omega_m^\alpha\right) b^m.$$

Conservation Laws

- Any Hamiltonian system

$$b_t^i = -\phi^{\beta\gamma}\psi_\beta^i \partial_x \psi_\gamma^s \frac{\delta \mathbf{H}}{\delta b^s}$$

possesses the conservation law of momentum:

$$P_t = \left[b^m \omega_m^\beta \psi_\beta^s \frac{\delta \mathbf{H}}{\delta b^s} - \frac{1}{3}\phi_{\beta\gamma}\omega_q^\beta b^q \psi_{km}^\gamma b^m \phi^{\alpha\delta}\psi_\alpha^k \left(\psi_\delta^s \frac{\delta \mathbf{H}}{\delta b^s}\right)_x - Q \right]_x,$$

where Q is a local expression of field variables b^k and all their higher derivatives due to well-known formula: $Q_x = \frac{\delta \mathbf{H}}{\delta b^s} b_x^s$ and

$$P = -\left(\frac{1}{3}\phi_{\beta\gamma}\omega_q^\beta \psi_{pm}^\gamma b_x^m + \frac{1}{2}\phi_{\beta\gamma}\omega_p^\beta \omega_q^\gamma\right) b^p b^q.$$

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- The WDVV hydrodynamic type systems are systems of conservation laws $a_t^i = (v^i(\mathbf{a}))_x$. This means that after a potential substitution $a^i = b_x^i$ we obtain the *nonlinear* system $b_t^i = v^i(\mathbf{b}_x)$ and the Hamiltonian can be obtained by solving the following system of PDEs:

$$v^i(\mathbf{b}_x) = -\phi^{\beta\gamma}\psi_\beta^i \partial_x \psi_\gamma^s \frac{\delta \mathbf{H}}{\delta b^s}.$$

  In the field variables b^k the Hamiltonian density becomes *local*, i.e. $\mathbf{H} = \int h(\mathbf{b}, \mathbf{b}_x)dx$.

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- Thus, in this case, one can obtain

$$h_t = -\left(\frac{\partial h}{\partial b^k_x}\phi^{\beta\gamma}\psi^k_\beta \partial_x \psi^s_\gamma \frac{\delta \mathbf{H}}{\delta b^s} + \frac{1}{2} g^{ks}(\mathbf{a}) \frac{\delta \mathbf{H}}{\delta b^k}\frac{\delta \mathbf{H}}{\delta b^s}\right)_x.$$

WDVV, N=4, Monge Metric

- **Theorem**: The metric $g_{in}(\mathbf{u})$ is transformed from the coordinates u^k to the coordinates $a^k(\mathbf{u})$ as

$$g_{ik}(\mathbf{a}) = \begin{pmatrix} (a^4)^2 & -2a^5 & 2a^4 & -(a^1a^4+a^3) & a^2 & 1 \\ -2a^5 & -2a^3 & a^2 & 0 & a^1 & 0 \\ 2a^4 & a^2 & 2 & -a^1 & 0 & 0 \\ -(a^1a^4+a^3) & 0 & -a^1 & (a^1)^2 & 0 & 0 \\ a^2 & a^1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The metric $g_{ik}(\mathbf{a})$ is a Monge metric satisfying Potemin system and it generates a Dubrovin–Novikov type third-order Hamiltonian operator $\hat{A}_2$ in canonical form

$$A_2^{ij} = \partial_x(g^{ij}\partial_x + c_k^{ij}a_x^k)\partial_x.$$

WDVV, N=4, Metric Decomposition

- **Theorem**: The Monge metric $g_{ij}(\mathbf{a})$ admits the decomposition, where

$$\psi_i^\gamma = \begin{pmatrix} 1 & a^5 & a^4 & 0 & 0 & 0 \\ 0 & a^3 & 0 & 1 & a^5 & 0 \\ 0 & -a^2 & 0 & 0 & -a^4 & 1 \\ 0 & 0 & -a^1 & 0 & a^3 & 0 \\ 0 & -a^1 & 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\phi_{\beta\gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

Here $\det \boldsymbol{\phi} = 1$, $\det \boldsymbol{\psi} = -(a^1)^2$.

WDVV, N=4, Second Hamiltonian Structure

- **Theorem**: So the Hamiltonian operator $\hat{A}_2$ can be rewritten in the simplified form

$$A_2^{ij} = \phi^{\beta\gamma}\partial_x\psi_\beta^i\partial_x\psi_\gamma^j\partial_x,$$

where inverse matrices are

$$\psi_\gamma^i = \frac{1}{a^1}\begin{pmatrix} a^1 & 0 & 0 & a^4 & a^5 & a^3a^4 - a^2a^5 \\ 0 & 0 & 0 & 0 & -1 & a^2 \\ 0 & 0 & 0 & -1 & 0 & -a^3 \\ 0 & a^1 & 0 & 0 & a^3 & a^1a^5 - a^2a^3 \\ 0 & 0 & 0 & 0 & 0 & -a^1 \\ 0 & 0 & a^1 & 0 & -a^2 & (a^2)^2 - a^1a^4 \end{pmatrix},$$

$$\phi^{\beta\gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

WDVV, N=4, Second Hamiltonian Structure

- **Theorem**: Each of the hydrodynamic type systems admits a Hamiltonian formulation by means of $\hat{A}_2$, with nonlocal Hamiltonian density $\tilde{h}_k(\mathbf{b}, \mathbf{b}_x)$, respectively

$$\tilde{h}_1 = -b^4 b^5 b_x^1 - b^5 b^2 b_x^2 + b^2 b^4 b_x^3 - b^2 b^6,$$
$$\tilde{h}_2 = -b^3 b^5 b_x^2 + b^4 b^3 b_x^3 + b^1 b^5 b_x^5 - b^3 b^6,$$

where $a^k = b_x^k$.

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where $a^k = b^k_x$. Both hydrodynamic type systems also have the same set of Casimirs $\mathbf{S}^k = \int s^k dx$ and a common momentum $\mathbf{P} = \int P dx$, where

$$s^1 = b^1, \quad s^2 = b^2, \quad s^3 = b^3, \quad s^4 = b^4 b^1_x,$$
$$s^5 = b^5 b^1_x + b^3 b^2_x, \quad s^6 = b^5 b^2_x + b^3 b^4_x + b^6;$$
$$P = -b^3 b^2 b^2_x - b^1 b^3 b^4_x + b^1 b^2 b^5_x - b^1 b^6 - (b^3)^2.$$

WDVV, N=4, Second Hamiltonian Structure

- **Remark**: Constant matrices η_1, η_2 are determined by the equations $\psi^{\gamma}_{m} v^m(\mathbf{b}_x) = \eta^{\gamma}_{1m} b^m_x$, $\psi^{\gamma}_{m} w^m(\mathbf{b}_x) = \eta^{\gamma}_{2m} b^m_x$ for both commuting systems $b^i_y = v^i(\mathbf{b}_x)$ and $b^i_z = w^i(\mathbf{b}_x)$. Here they are

$$\eta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\eta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

- **Theorem**: The operators $\hat{A}_1, \hat{A}_2$ form a commuting pair, hence the