El's Nonlocal Kinetic equation and its Hydrodynamic Reductions

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The Talk is based on joint works with my friends and colleagues:
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Conjugate Curvilinear Coordinate Nets

We start from the linear system

$$\partial_i H_k = \beta_{ik} H_i, \quad i \neq k,$$

where H_i and rotation coefficients β_{ik} depend on N independent variables r^k.

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The compatibility conditions $\partial_i(\partial_j H_k) = \partial_j(\partial_i H_k)$ for every triad of distinct indices lead to the Lame system

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This system also can be obtained from the compatibility conditions $\partial_i(\partial_j \psi_k) = \partial_j(\partial_i \psi_k)$ for every triad of distinct indices, where the vector function ψ_i satisfies the adjoint linear system

$$\partial_i \psi_k = \beta_{ki} \psi_i, \quad i \neq k.$$

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The Lame system is a 3D integrable system. Indeed, it is easy to see for every three distinct indices:

$$\partial_{r^1}\beta_{23} = \beta_{21}\beta_{13}, \quad \partial_{r^1}\beta_{32} = \beta_{31}\beta_{12},$$
$$\partial_{r^2}\beta_{13} = \beta_{12}\beta_{23}, \quad \partial_{r^2}\beta_{31} = \beta_{32}\beta_{21},$$
$$\partial_{r^3}\beta_{12} = \beta_{13}\beta_{32}, \quad \partial_{r^3}\beta_{21} = \beta_{23}\beta_{31}.$$

Conjugate Egorov Curvilinear Coordinate Nets

The Lame system

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}, \quad i \neq j \neq k,$$

is determined by two alternative Lax representations

$$\partial_i H_k = \beta_{ik} H_i, \quad \partial_i \psi_k = \beta_{ki}\psi_i, \quad i \neq k.$$

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Now we consider the 3D symmetric reduction

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In this case, both Lax representations coincide with each other (i.e. $\psi_i \equiv H_i$). This means, symmetric rotation coefficients β_{ik} of conjugate Egorov curvilinear nets satisfy the Lame–Egorov system

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}, \quad i \neq j \neq k; \quad \beta_{ik} = \beta_{ki}, \quad i \neq k.$$

Again in the three dimensional case we have only three equations (instead of six equations without the symmetric reduction)

$$\partial_{r^1}\beta_{23} = \beta_{21}\beta_{13}, \quad \partial_{r^2}\beta_{13} = \beta_{12}\beta_{23}, \quad \partial_{r^3}\beta_{12} = \beta_{13}\beta_{32}.$$

Conjugate Linearly Degenerate Curvilinear Coordinate Nets

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Now we consider the reduction

$$\partial_i \beta_{kk} = \beta_{ik}\beta_{ki}, \quad \partial_k \ln \beta_{ik} = \beta_{kk}, \quad i \neq k.$$

In this case the Lame system is the so called *Darboux solvable*. All rotation coefficients $\beta_{ik}(\mathbf{r})$ can be found explicitly. This case is known in the theory of *semi-Hamiltonian* hydrodynamic type systems as *linearly degenerate*. See: E.V. Ferapontov (1991).

Conjugate Temple Curvilinear Coordinate Nets

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Obviously, in this case the Lame system is *Darboux solvable* as well as in the linearly degenerate case selected by the first reduction

$$\partial_i \beta_{kk} = \beta_{ik} \beta_{ki}, \quad \partial_k \ln \beta_{ik} = \beta_{kk}, \quad i \neq k.$$

This case is known in the theory of *semi-Hamiltonian* hydrodynamic type systems as hydrodynamic type systems of *Temple's class*.

Conjugate Egorov Linearly Degenerate Curvilinear Coordinate Nets

The Lame–Egorov system

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}, \quad i \neq j \neq k; \quad \beta_{ik} = \beta_{ki}, \quad i \neq k$$

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Now we consider the first reduction (linearly degeneracy)

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This particular class of conjugate curvilinear coordinate nets is not yet described.

Conjugate Egorov–Temple Curvilinear Coordinate Nets

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Now we consider the second reduction (Temple's class)

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Now we consider both reductions simultaneously

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Then all above nonlinear equations can be written in the compact form (no restrictions on coincided indices!)

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Then all above nonlinear equations can be written in the compact form (no restrictions on coincided indices!)

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}.$$

This particular class of conjugate curvilinear coordinate nets is described below.

A Complete Description of Conjugate Temple Linearly Degenerate Curvilinear Coordinate Nets

Let us introduce the $N \times N$ matrix $\hat{\epsilon}$ with diagonal entries $r^1, \ldots, r^N$ (so that $\epsilon^{ii} = r^i$) and off-diagonal entries $\epsilon^{ik} =$const, $k \neq i$.

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Then one can see that the coefficients β_{ik} of this matrix $\hat{\beta}$ satisfy the nonlinear system

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$$\partial_i \beta_{jk} = \beta_{ji} \beta_{ik}.$$

Indeed, differentiation of $\beta_{km} \epsilon^{mi} = -\delta^i_k$ with respect to any variable r^j implies

$$0 = \beta_{im} \partial_j \epsilon^{mk} + (\partial_j \beta_{im}) \epsilon^{mk}.$$

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Multiplying by the matrix $\hat{\beta}$, one can obtain

$$0 = \beta_{im} (\partial_j \epsilon^{mk}) \beta_{ks} + (\partial_j \beta_{im}) \epsilon^{mk} \beta_{ks}.$$

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Multiplying by the matrix $\hat{\beta}$, one can obtain

$$0 = \beta_{im} (\partial_j \epsilon^{mk}) \beta_{ks} + (\partial_j \beta_{im}) \epsilon^{mk} \beta_{ks}.$$

Taking into account $\beta_{km}\epsilon^{mi} = -\delta^i_k$, finally we have

$$\partial_j \beta_{is} = \beta_{im} (\partial_j \epsilon^{mk}) \beta_{ks} \equiv \beta_{ij} \beta_{js}.$$

A Complete Description of Conjugate Temple Linearly Degenerate Curvilinear Coordinate Nets

Vice Versa: Assume that the coefficients β_{ik} of the $N \times N$ matrix $\hat{\boldsymbol{\beta}}$ satisfy the nonlinear system

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Define another matrix $\hat{\boldsymbol{\epsilon}} = -\hat{\boldsymbol{\beta}}^{-1}$.
Multiplying both sides of the above nonlinear system $\partial_i \beta_{jk} = \beta_{ji} \beta_{ik}$ by ϵ^{pj} from the left, and by ϵ^{kq} from the right, we obtain

$$\epsilon^{pj} (\partial_i \beta_{jk}) \epsilon^{kq} = (\epsilon^{pj} \beta_{ji})(\beta_{ik} \epsilon^{kq}) = \delta_i^p \delta_i^q.$$

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$$\epsilon^{pj} (\partial_i \beta_{jk}) \epsilon^{kq} = (\epsilon^{pj} \beta_{ji})(\beta_{ik} \epsilon^{kq}) = \delta_i^p \delta_i^q.$$

Again, taking into account $\beta_{km} \epsilon^{mi} = -\delta_k^i$, the l.h.s. of the above expression becomes

$$\epsilon^{pj} (\partial_i \beta_{jk}) \epsilon^{kq} = \epsilon^{pj} [\partial_i (\beta_{jk} \epsilon^{kq}) - \beta_{jk} \partial_i \epsilon^{kq}] = [\partial_i (\epsilon^{pj} \beta_{jk}) - (\partial_i \epsilon^{pj}) \beta_{jk}] \epsilon^{kq} = \partial_i \epsilon^{pq}$$

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$$\epsilon^{pj} (\partial_i \beta_{jk}) \epsilon^{kq} = \epsilon^{pj} [\partial_i (\beta_{jk} \epsilon^{kq}) - \beta_{jk} \partial_i \epsilon^{kq}] = [\partial_i (\epsilon^{pj} \beta_{jk}) - (\partial_i \epsilon^{pj}) \beta_{jk}] \epsilon^{kq} = \partial_i \epsilon^{pq}$$

This means: the general solution of the system $\partial_i \epsilon^{pq} = \delta_i^p \delta_i^q$ is determined by the $N \times N$ matrix $\hat{\boldsymbol{\epsilon}}$ with diagonal entries $r^1, \dots, r^N$ (so that $\epsilon^{ii} = r^i$) and off-diagonal entries $\epsilon^{ik} =$const, $k \neq i$.

Orthogonal Curvilinear Coordinate Nets

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is determined by two alternative Lax representations

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Now we consider the 3D reduction (here $\psi_{i,i} \equiv \partial_i \psi_i$)

$$H_i = \psi_{i,i} + \sum_{m \neq i} \beta_{mi}\psi_m.$$

The above Lax representations are connected to each other via this differential substitution of first order, if rotation coefficients β_{ik} satisfy to extra set of nonlinear equations (the Gauss system)

$$\partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{m \neq i,k} \beta_{mi}\beta_{mk} = 0, \quad i \neq k.$$

Thus, the Darboux–Lame system is

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}, \quad i \neq j \neq k; \quad \partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{m \neq i,k} \beta_{mi}\beta_{mk} = 0, \quad i \neq k.$$

Egorov Orthogonal Curvilinear Coordinate Nets

The Darboux–Lame system

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is determined by the Lax representation (here $\psi_{i,i} \equiv \partial_i \psi_i$)

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is determined by the Lax representation (here $\psi_{i,i} \equiv \partial_i\psi_i$)

$$\partial_i H_k = \beta_{ik}H_i, \quad \partial_i\psi_k = \beta_{ki}\psi_i, \quad i \neq k; \quad H_i = \psi_{i,i} + \sum_{m\neq i}\beta_{mi}\psi_m.$$

Now we consider the 3D symmetric reduction $\beta_{ik} = \beta_{ki}$.

Egorov Orthogonal Curvilinear Coordinate Nets

The Darboux–Lame system

$$\partial_i\beta_{jk}=\beta_{ji}\beta_{ik},\quad i\neq j\neq k;\quad \partial_i\beta_{ik}+\partial_k\beta_{ki}+\sum_{m\neq i,k}\beta_{mi}\beta_{mk}=0,\quad i\neq k,$$

is determined by the Lax representation (here $\psi_{i,i}\equiv\partial_i\psi_i$)

$$\partial_iH_k=\beta_{ik}H_i,\quad \partial_i\psi_k=\beta_{ki}\psi_i,\quad i\neq k;\quad H_i=\psi_{i,i}+\sum_{m\neq i}\beta_{mi}\psi_m.$$

Now we consider the 3D symmetric reduction $\beta_{ik}=\beta_{ki}$.
Then the Gauss system

$$\partial_i\beta_{ik}+\partial_k\beta_{ki}+\sum_{m\neq i,k}\beta_{mi}\beta_{mk}=0,\quad i\neq k,$$

reduces to the form $\delta\beta_{ik}=0$, while the differential substitution

$$H_i=\psi_{i,i}+\sum_{m\neq i}\beta_{mi}\psi_m$$

becomes $H_i=\delta\psi_i$, where $\delta=\Sigma\partial/\partial r^m$ is a shift symmetry operator.

Frobenius Manifolds

The Darboux–Lame–Egorov system

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}, \quad i \neq j \neq k; \quad \delta\beta_{ik} = 0,$$

is determined by the Lax representation (here $\psi_{i,i} \equiv \partial_i \psi_i$)

$$\partial_i H_k = \beta_{ik} H_i, \quad i \neq k; \quad \lambda H_i = \delta H_i,$$

where λ is a spectral parameter.

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where λ is a spectral parameter. This is 2D integrable system, obtained on the intersection of two 3D reductions (Egorov case + Orthogonal case). The concept of a Frobenius manifold appears in extension of the Darboux–Lame–Egorov system:

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}, \quad i \neq j \neq k; \quad \delta\beta_{ik} = 0, \quad \hat{R}\beta_{ik} = -\beta_{ik},$$

where $\hat{R} = \Sigma r^m \partial/\partial r^m$ is the Euler (scaling) symmetry operator.

A Simplest Nontrivial Example of Egorov Curvilinear Coordinate Nets

Let us introduce the $N \times N$ symmetric matrix $\hat{\epsilon}$ with diagonal entries $r^1, \ldots, r^N$ (so that $\epsilon^{ii} = r^i$) and off-diagonal entries $\epsilon^{ik} = \epsilon^{ki} =$const, $k \neq i$.

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We remind that the Lame system is a 3D integrable system.

Alternative Approach. Exceptional Lax Representations

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One can select any pair of particular solutions $\bar{H}_i$ and $\tilde{H}_i$ of the first linear system

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Here we remind that independent variables are r^k. So, $\partial_k \equiv \partial / \partial r^k$.
Now we introduce an N component hydrodynamic type system

$$r_t^i = \mu^i(\mathbf{r}) r_x^i,$$

whose characteristic velocities

$$\mu^i(\mathbf{r}) = \frac{\tilde{H}_i}{\bar{H}_i}.$$

This hydrodynamic type system is integrable by Tsarev's Generalised Hodograph Method. In this construction: Riemann invariants r^k are functions of two independent variables x and t only.

Commuting Flows

Integrable N component hydrodynamic type system

$$r_t^i = \mu^i(\mathbf{r}) r_x^i$$

has infinitely many commuting flows (τ is the so called group parameter in the Lie group analysis, or an auxiliary time variable)

$$r_\tau^i = \zeta^i(\mathbf{r}) r_x^i.$$

This means, that the Riemann invariants r^i no longer depend on **two** independent variables x and t only. Now, the Riemann invariants r^i depend on three independent variables x, t, τ simultaneously.

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This means, that the Riemann invariants $r^i(x, t, \tau)$ solve two N component hydrodynamic type systems

$$r^i_t = \mu^i(\mathbf{r})r^i_x, \quad r^i_\tau = \zeta^i(\mathbf{r})r^i_x,$$

where the time variable τ is hidden in the first hydrodynamic type system, while the time variable t is hidden in the second hydrodynamic type system. Then both hydrodynamic type systems must commute with each other.

Commuting Flows

The compatibility conditions $(r_t^i)_\tau = (r_\tau^i)_t$ lead to the Tsarev conditions

$$\frac{\partial_k \mu^i}{\mu^k - \mu^i} = \frac{\partial_k \zeta^i}{\zeta^k - \zeta^i}, \quad i \neq k.$$

Taking into account the definition of the Lame coefficients

$$\partial_k \ln \bar{H}_i = \frac{\partial_k \mu^i}{\mu^k - \mu^i}, \quad i \neq k,$$

the Tsarev conditions show that both commuting hydrodynamic type systems

$$r_t^i = \mu^i(\mathbf{r}) r_x^i, \quad r_\tau^i = \zeta^i(\mathbf{r}) r_x^i$$

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Any diagonalisable hydrodynamic type system

$$r_t^i = \mu^i(\mathbf{r}) r_x^i, \quad i = 1, 2, ..., N$$

is integrable by Tsarev's Generalised Hodograph Method

$$x + \mu^i(\mathbf{r}) t = \zeta^i(\mathbf{r}),$$

if and only if the integrability condition (here $\partial_k \equiv \partial / \partial r^k$)

$$\partial_j \frac{\partial_k \mu^i}{\mu^k - \mu^i} = \partial_k \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j \neq k$$

is fulfilled. Here we remind that diagonal metric coefficients $g_{kk}(\mathbf{r}) = \bar{H}_k^2$ are determined by

$$\partial_k \ln \bar{H}_i = \frac{\partial_k \mu^i}{\mu^k - \mu^i}, \quad i \neq k,$$

while $\zeta^i(\mathbf{r})$ satisfy to the linear system

$$\partial_k \zeta^i = \frac{\partial_k \mu^i}{\mu^k - \mu^i} (\zeta^k - \zeta^i), \quad i \neq k.$$

El's Nonlocal Kinetic Equation

El's integro-differential kinetic equation for dense soliton gas (2003)

$$f_t + (sf)_x = 0,$$

$$s(\eta) = S(\eta) + \int_0^\infty G(\mu,\eta) f(\mu)[s(\mu) - s(\eta)]\, d\mu,$$

where $f(\eta) = f(\eta, x, t)$ is a distribution function and $s(\eta) = s(\eta, x, t)$ is the associated transport velocity. Here the variable η is the spectral parameter in the Lax pair; the function $S(\eta)$ (free soliton velocity) and the kernel $G(\mu,\eta)$ (phase shift due to pairwise soliton collisions) are independent of x and t. The kernel $G(\mu,\eta)$ is assumed to be symmetric: $G(\mu,\eta) = G(\eta,\mu)$. This system describes the evolution of a dense soliton gas and represents a broad generalisation of Zakharov's kinetic equation for rarefied soliton gas. In this case

$$S(\eta) = 4\eta^2, \qquad G(\mu,\eta) = \frac{1}{\eta\mu} \log\left|\frac{\eta-\mu}{\eta+\mu}\right|,$$

the above system was derived by G. El as thermodynamic limit of the KdV Whitham equations.

El's Nonlocal Kinetic Equation. Zakharov Approximation

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$$s(\eta) = S(\eta) + \int_0^\infty G(\mu,\eta) f(\mu)[s(\mu) - s(\eta)]\, d\mu.$$

Taking into account the dependence $S(\eta) = 4\eta^2$, the integral equation

$$s(\eta) = 4\eta^2 + \int\limits_0^\infty G(\mu,\eta) f(\mu)[s(\mu) - s(\eta)]\, d\mu$$

in the zero-order approximation is $s(\eta) = 4\eta^2$ only. This means, in the first-order approximation, one can obtain

$$s(\eta) = 4\eta^2 + \int\limits_0^\infty G(\mu,\eta) f(\mu)(4\mu^2 - 4\eta^2) d\mu.$$

It was exactly equation derived by V.E. Zakharov for rarefied gas in 1971.

Hydrodynamic Reductions. Dirac Delta-Functional Ansatz

Under a delta-functional ansatz (an **iso-spectral** case, 2010, G.A. El, A.M. Kamchatnov, MVP, S.A. Zykov),

$$f(\eta, x, t) = \sum_{i=1}^{n} u^i(x, t)\ \delta(\eta - \eta^i),$$

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where v^i can be recovered from the linear system (here $\xi^i = -S(\eta^i)$)

$$v^i = \xi^i + \sum_{m \neq i} \epsilon^{mi} u^m (v^m - v^i), \qquad \epsilon^{ki} = G(\eta^k, \eta^i),\ k \neq i.$$

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where velocities v^i can be expressed in terms of Riemann invariants as follows. Let us introduce the $N \times N$ matrix $\hat{\epsilon}$ with diagonal entries $r^1, \ldots, r^N$ (so that $\epsilon^{ii} = r^i$) and off-diagonal entries $\epsilon^{ik} = G(\eta^i, \eta^k),\ k \neq i$.

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Define another symmetric matrix $\hat{\beta} = -\hat{\epsilon}^{-1}$.

Tsarev's Generalised Hodograph Method

Denote β_{ik} the matrix elements of the matrix $\hat{\beta}$ (indices i and k are allowed to coincide). Then we obtain the following formulae for u^i, v^i:

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$$x + \xi_i t = P_i(r^i) - r^i P_i'(r^i) - \sum_{m \neq i} \epsilon^{mi} P_m'(r^m), \quad i = 1, 2, ..., N,$$

where $P_i(r^i)$, $i = 1, \ldots, N$, are arbitrary functions.

Tsarev's Generalised Hodograph Method

Under the re-parametrization

$$P_k''(\xi) = -\frac{\phi_k(\xi)}{f(\xi)}$$

the generalized hodograph solution

$$x + \xi_i t = P_i(r^i) - r^i P_i'(r^i) - \sum_{m \neq i} \epsilon^{mi} P_m'(r^m), \quad i = 1, 2, ..., N,$$

becomes

$$x + \xi_i t = \int^{r^i} \frac{\xi \phi_i(\xi)}{f(\xi)} d\xi + \sum_{m \neq i} \epsilon^{mi} \int^{r^m} \frac{\phi_m(\xi)}{f(\xi)} d\xi \, .$$

Tsarev's Generalised Hodograph Method

Now we consider the particular choice of $f(\xi)$ defined as $f(\xi) = \sqrt{R_K(\xi)}$, where

$$R_K(\xi) = \prod_{m=1}^{K} (\xi - E_m),$$

and $E_1 < E_2 < \cdots < E_K$ are real constants ($K = 2N + 1$ and $K = 2N + 2$ for odd and even number of branch points of this hyperelliptic curve of a genus N); and $\phi_k(\xi)$ being arbitrary polynomials in ξ of degrees less than N.

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Then the generalized hodograph solution

$$x + \xi_i t = \int^{r^i} \frac{\xi \phi_i(\xi)}{f(\xi)} d\xi + \sum_{m \neq i} \epsilon^{mi} \int^{r^m} \frac{\phi_m(\xi)}{f(\xi)} d\xi \, ,$$

describes quasiperiodic solutions of the form

$$x + \xi_i t = \int^{r^i} \frac{\xi \phi_i(\xi) d\xi}{\sqrt{R_K(\xi)}} + \sum_{m \neq i} \epsilon^{mi} \int^{r^m} \frac{\phi_m(\xi) d\xi}{\sqrt{R_K(\xi)}}, \quad i = 1, 2, ..., N.$$

A Nijenhuis tensor

Recall that, given an affinor V^i_k, its Haantjes tensor is defined by the formula

$$H^i_{jk} = N^i_{pr} V^p_j V^r_k - N^p_{jr} V^i_p V^r_k - N^p_{rk} V^i_p V^r_j + N^p_{jk} V^i_r V^r_p,$$

where

$$N^i_{jk} = V^p_j \partial_p V^i_k - V^p_k \partial_p V^i_j - V^i_p (\partial_j V^p_k - \partial_k V^p_j)$$

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where

$$N^i_{jk} = V^p_j \partial_p V^i_k - V^p_k \partial_p V^i_j - V^i_p (\partial_j V^p_k - \partial_k V^p_j)$$

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In a generic case all characteristic velocities μ^k are *pairwise distinct*. If all components of a Nijenhuis tensor vanish, then corresponding hydrodynamic type system

$$u^i_t = V^i_k(\mathbf{u}) u^k_x$$

can be reduced to the totally decoupled form

$$\tilde{u}^i_t = \mu^i(\tilde{u}^i) \tilde{u}^i_x$$

by an appropriate invertible point transformation $\tilde{u}^k(\mathbf{u})$.

A Haantjes tensor

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while a Nijenhuis tensor is

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while a Nijenhuis tensor is

$$N^i_{jk} = V^p_j \partial_p V^i_k - V^p_k \partial_p V^i_j - V^i_p (\partial_j V^p_k - \partial_k V^p_j).$$

In a generic case all characteristic velocities μ^k are *pairwise distinct*. If all components of a Haantjes tensor vanish, then corresponding hydrodynamic type system

$$u^i_t = V^i_k(\mathbf{u}) u^k_x$$

can be diagonalised, i.e. rewritten in the Riemann invariants

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A Haantjes tensor and Integrable Hydrodynamic Type Systems

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The **Statement**: *If a hydrodynamic type system is integrable by Tsarev's Generalised Hodograph Method, then all components of a Haantjes tensor vanish. Then this hydrodynamic type system can be reduced to a* **block-diagonal** *structure by an appropriate invertible point transformation* $\tilde{u}^k(\mathbf{u})$.

Hydrodynamic Reductions. Dirac Delta-Functional Ansatz

Under a delta-functional ansatz (a **non-isospectral** case, 2012, G.A. El, V.B. Taranov, MVP),

$$f(\eta, x, t) = \sum_{i=1}^{N} u^i(x, t)\ \delta(\eta - \eta^i(x, t)),$$

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where v^i can be recovered from the linear system (here $\xi^i = -S(\eta^i)$)

$$v^i = \xi^i + \sum_{m \neq i} \epsilon^{mi} u^m (v^m - v^i), \qquad \epsilon^{ki} = G(\eta^k, \eta^i),\ k \neq i.$$

Block-Diagonal Hydrodynamic Type Systems

Introducing new field variables

$$r^i = -\frac{1}{u^i}\left(1 + \sum_{m \neq i} \epsilon^{mi} u^m\right),$$

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Block-Diagonal Hydrodynamic Type Systems

Now we study integrability aspects of quasilinear systems

$$u_t^i = V_k^i(\mathbf{u}) u_x^k,$$

whose matrix V consists of N Jordan blocks of size 2×2:

$$\begin{aligned} r_t^i &= v^i r_x^i + p^i \eta_x^i, \\ \eta_t^i &= v^i \eta_x^i, \end{aligned}$$

$i = 1, \ldots, N$, where the coefficients $v^i(r, \eta)$ and $p^i(r, \eta)$ are functions of the N dependent variables $r = (r^1, \ldots, r^N)$ and N dependent variables $\eta = (\eta^1, \ldots, \eta^N)$.

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Their commuting flows $u_y^i = W_k^i(\mathbf{u})u_x^k$ are in the same form (2021, E.V. Ferapontov, MVP)

$$\begin{aligned} r_y^i &= w^i r_x^i + q^i \eta_x^i, \\ \eta_y^i &= w^i \eta_x^i. \end{aligned}$$

Then unknown expressions $w^i(\mathbf{r}, \boldsymbol{\eta})$, $q^i(\mathbf{r}, \boldsymbol{\eta})$ can be found from the compatibility conditions $(r_y^i)_t = (r_t^i)_y$, $(\eta_y^i)_t = (\eta_t^i)_y$, $i = 1, 2, ..., N$.

Block-Diagonal Hydrodynamic Type Systems

Indeed, the compatibility conditions

$$(r_y^i)_t = (r_t^i)_y, \quad (\eta_y^i)_t = (\eta_t^i)_y, \quad i = 1, 2, ..., N$$

lead to the set of equations

$$w_{r^i}^i = a_i q^i, \quad w_{\eta^i}^i = b_i q^i + q_{r^i}^i,$$

where we denote

$$a_i = \frac{v_{r^i}^i}{p^i}, \quad b_i = \frac{v_{\eta^i}^i - p_{r^i}^i}{p^i}.$$

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where we denote

$$a_i = \frac{v^i_{r^i}}{p^i}, \quad b_i = \frac{v^i_{\eta^i} - p^i_{r^i}}{p^i}.$$

$$w^i_{r^j} = a_{ij}(w^j - w^i), \quad w^i_{\eta^j} = b_{ij}(w^j - w^i) + a_{ij} q^j,$$

$$q^i_{r^j} = c_{ij}(w^j - w^i) - a_{ij} q^i, \quad q^i_{\eta^j} = d_{ij}(w^j - w^i) + c_{ij} q^j - b_{ij} q^i,$$

where we denote

$$a_{ij} = \frac{v^i_{r^j}}{v^j - v^i}, \quad b_{ij} = \frac{v^i_{\eta^j} - a_{ij} p^j}{v^j - v^i}, \quad c_{ij} = \frac{p^i_{r^j} + a_{ij} p^i}{v^j - v^i}, \quad d_{ij} = \frac{p^i_{\eta^j} + b_{ij} p^i - c_{ij} p^j}{v^j - v^i}.$$

Integrability Conditions I

The list of integrability conditions for every pair of distinct indices is

$$a_{i,r^j} = 0, \qquad a_{ij,r^i} = a_{ij}a_{ji} + a_i c_{ij};$$

$$a_{i,\eta^j} = 0, \qquad b_{ij,r^i} = b_{ij}a_{ji} + a_{ij}c_{ji} + a_i d_{ij};$$

$$b_{i,r^j} = 2a_{ij}a_{ji} + 2a_i c_{ij},$$

$$a_{ij,\eta^i} = a_{ij}b_{ji} - c_{ij}a_{ji} + b_i c_{ij} + c_{ij,r^i};$$

$$b_{i,\eta^j} = 2a_{ij}c_{ji} + 2b_{ij}a_{ji} + 2a_i d_{ij},$$

$$b_{ij,\eta^i} = b_{ij}b_{ji} + a_{ij}d_{ji} - d_{ij}a_{ji} - c_{ij}c_{ji} + b_i d_{ij} + d_{ij,r^i};$$

$$a_{ij,r^j} = b_j a_{ij} - a_j b_{ij} - a_{ij}^2, \qquad a_{ij,\eta^j} = b_{ij,r^j};$$

$$c_{ij,r^j} = b_j c_{ij} - a_j d_{ij} - 2a_{ij}c_{ij}, \qquad c_{ij,\eta^j} = d_{ij,r^j}.$$

Integrability Conditions II

The list of integrability conditions for every triad of distinct indices is

$$a_{ij,r^k} = a_{ij}a_{jk} + a_{ik}a_{kj} - a_{ij}a_{ik}.$$

$$a_{ij,\eta^k} = a_{ij}b_{jk} + a_{ik}c_{kj} + b_{ik}a_{kj} - a_{ij}b_{ik},$$
$$b_{ij,r^k} = b_{ij}a_{jk} + a_{ik}b_{kj} + a_{ij}c_{jk} - a_{ik}b_{ij}.$$

$$b_{ij,\eta^k} = a_{ij}d_{jk} + a_{ik}d_{kj} + b_{ij}b_{jk} + b_{ik}b_{kj} - b_{ij}b_{ik}.$$

$$c_{ij,r^k} = c_{ij}a_{jk} + c_{ik}a_{kj} - c_{ij}a_{ik} - c_{ik}a_{ij}.$$

$$c_{ij,\eta^k} = c_{ij}b_{jk} + c_{ik}c_{kj} + d_{ik}a_{kj} - a_{ij}d_{ik} - c_{ij}b_{ik},$$
$$d_{ij,r^k} = d_{ij}a_{jk} + c_{ij}c_{jk} + c_{ik}b_{kj} - a_{ik}d_{ij} - c_{ik}b_{ij}.$$

$$d_{ij,\eta^k} = c_{ij}d_{jk} + c_{ik}d_{kj} + d_{ij}b_{jk} + d_{ik}b_{kj} - b_{ij}d_{ik} - b_{ik}d_{ij}.$$

Commuting Flows

The block-diagonal system

$$r_t^i = v^i r_x^i + p^i \eta_x^i, \quad \eta_t^i = v^i \eta_x^i,$$

where

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \ v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}, \ p^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon_{,\eta^i}^{mi} (v^m - v^i) u^m + (\xi^i)' \right),$$

possesses infinitely many commuting block-diagonal flows

$$r_y^i = w^i r_x^i + q^i \eta_x^i, \quad \eta_y^i = w^i \eta_x^i,$$

where

$$w^i = \frac{1}{u^i} \sum_{m=1}^{N} \varphi^m \beta_{mi}, \qquad q^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon_{,\eta^i}^{mi} (w^m - w^i) u^m - r^i \mu^i + \varphi_{,\eta^i}^i \right).$$

Here $\mu^i(\eta^i)$ are N arbitrary functions of one variable and the functions $\varphi^i(\eta^1, \ldots, \eta^N)$ satisfy the relations $\partial_{\eta^k} \varphi^i = \epsilon^{ki} \mu^k, \ k \neq i$. The general commuting flow depends on $2N$ arbitrary functions of one variable: N functions $\mu^i(\eta^i)$, plus extra N functions coming from φ^i.

Conservation Laws

Conservation laws $h_t = g_x$ provide an alternative way to derive integrability conditions for the block-diagonal system

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x.$$

Their existence leads to a system of second-order linear PDEs

$$h_{r^i r^i} = b_i h_{r^i} - a_i h_{\eta^i}, \quad h_{r^i \eta^j} = a_{ji} h_{\eta^j} + c_{ji} h_{r^j} + b_{ij} h_{r^i},$$

$$h_{r^i r^j} = a_{ij} h_{r^i} + a_{ji} h_{r^j}, \quad h_{\eta^i \eta^j} = d_{ij} h_{r^i} + d_{ji} h_{r^j} + b_{ij} h_{\eta^i} + b_{ji} h_{\eta^j},$$

where $g_{r^i} = v^i h_{r^i}$, $g_{\eta^i} = p^i h_{r^i} + v^i h_{\eta^i}$.

The general conservation law has the form ($\sigma^i(\eta^i)$ are arbitrary functions)

$$\left(\sum_{m=1}^N u^m \psi^m(\eta) + \sum_{m=1}^N \sigma^m(\eta^m)\right)_t = \left(\sum_{m=1}^N u^m v^m \psi^m(\eta) + \sum_{m=1}^N \tau^m(\eta^m)\right)_x,$$

where $(\tau^i)' = (\sigma^i)'\xi^i$ and $\psi^i_{,\eta^k} = (\sigma^j)'\epsilon^{ik}$, $k \neq i$. This general conservation law depends on $2N$ arbitrary functions of one variable: N functions $\sigma^i(\eta^i)$, plus extra N functions coming from ψ^i.

Tsarev's Generalised Hodograph Method

If the hydrodynamic type system $u_t = V(u)u_x$ has a commuting flow $u_y = W(u)u_x$, where $V(u)$ and $W(u)$ are $N \times N$ matrices (the commutativity conditions $u_{ty} = u_{yt}$ impose differential constraints on V and W),

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$$W(u) = I\,x + V(u)\,t,$$

where I is the $N \times N$ identity matrix, defines an implicit solution $u(x,t)$. Note that, due to the commutativity conditions, only N out of the above N^2 relations will be functionally independent.

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$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x,$$
$$r^i_y = w^i r^i_x + q^i \eta^i_x, \quad \eta^i_y = w^i \eta^i_x,$$

the hodograph formula becomes

$$w^i(r,\eta) = x + v^i(r,\eta)\,t, \qquad q^i(r,\eta) = p^i(r,\eta)\,t,$$

which is a system of $2N$ implicit relations for the $2N$ dependent variables.

Tsarev's Generalised Hodograph Method

Denote β_{ik} the matrix elements of $\hat{\beta}$ (indices i and k are allowed to coincide). Then we obtain the following formulae for u^i, v^i and p^i:

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \; v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}, \; p^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon^{mi}_{,\eta^i} (v^m - v^i) u^m + (\xi^i)' \right).$$

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$$r^i = \frac{\varphi^i_{,\eta^i} - (\xi^i)'\, t}{\mu^i}, \qquad \varphi^i(\eta^1, \ldots, \eta^N) = x + \xi^i(\eta^i)\, t;$$

where $\mu^i(\eta^i)$ are arbitrary functions of their arguments and the functions $\varphi^i(\eta^1, \ldots, \eta^N)$ satisfy the relations $\varphi^i_{,\eta^k} = \epsilon^{ki}(\eta^i, \eta^k)\, \mu^k(\eta^k)$, $i \neq k$. The last N above equations define $\eta^i(x,t)$ as implicit functions of x and t; then the first N equations define $r^i(x,t)$ explicitly.

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B.A. Dubrovin considered remarkable WDVV associativity equations, whose solutions determine families (primary flows) of commuting Hamiltonian Egorov hydrodynamic type systems integrable by Tsarev's Generalised Hodograph Method.

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Our Claim is: *Dubrovin's Program can be easily extended to a non-diagonalisable case due to existence of a special coordinate system, where velocity matrices can be reduced to a block-diagonal form*.

For instance, in the three-component case, one has **three** options: three distinct characteristic velocities; two distinct characteristic velocities; one common characteristic velocity. In the four-component case, we have already **five** options: four distinct characteristic velocities; one Jordan block 2x2 and three distinct characteristic velocities; two Jordan blocks 2x2 and two distinct characteristic velocities; one Jordan block 3x3 and two distinct characteristic velocities; one Jordan block 4x4 and one common characteristic velocity only.

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