# On a class of integrable lattices in 3D

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# Degenerate cutoff conditions for the integrable lattices

Let us explain the phenomenon with the famous Volterra chain

$$u_{n,t} = u_n(u_{n+1} - u_{n-1}), \quad -\infty < n < +\infty$$

We can reduce it to a finite system of the ordinary differential equations by several ways, preserving integrability.

$$\begin{aligned} &u_0 = 0, \\ &u_{n,t} = u_n(u_{n+1} - u_{n-1}), \quad 1 < n < N, \\ &u_{N+1} = 0, \end{aligned}$$

The obtained system is solved explicitly in terms of the elementary functions. Another kind of BC $u_0 = -u_1$ and $u_{N+1} = -u_N$. Solutions to that case is given in terms of the hyperelliptic functions. The reason: the BC $u_0 = 0$ is compatible with all higher symmetries of the chain, while the case $u_0 = -u_1$ is consistent only with some of the symmetries (half of the set).

Toda lattice with the BC $a_0 = a_n = 0$ (consistent with all higher symmetries)

$$a_{k,t} = a_k(b_k - b_{k+1}),$$
$$b_{n,t} = 2(a_{k-1}^2 - a_k^2), \quad 1 < n < N-1,$$

is solved in elementary functions (J. Moser, 1975). We change the variables

$$(a_1, a_2, ...a_{n-1}; b_1, ...b_n) \Rightarrow (\lambda_1, ..., \lambda_n; r_1, ...r_n)$$

due to

$$f(\lambda) = \sum \frac{r_k^2}{\lambda - \lambda_k} = \frac{1}{\lambda - b_n - \frac{a_{n-1}}{\lambda - b_{n-1} - ... - \frac{a_1}{\lambda - b_1}}}$$

Here $Imf(\lambda) > 0$ iff $Im\lambda > 0$ and $\lambda f(\lambda) \to 1$ for $\lambda \to \infty$. If $a_j > 0$, $b_j$ are real, $\lambda_i$ are real and pairwise different, $r_k$ are real and $\sum r_k^2 = 1$ then the change is invertible. The new equations look like

The new equations look like

$$\lambda_{k,t} = 0, \quad r_{k,t} = (\lambda_k - \sum \lambda_j r_j^2) r_k.$$

A solution to the system

$$\lambda_k = const, \quad r_k^2 = \frac{r_k^2(0) e^{2\lambda_k t}}{\sum r_j^2(0) e^{2\lambda_j t}}$$

It remains to find $a_j$ and $b_j$ due to Euclidean algorithm. Apparently any integrable lattice of Toda and Volterra type admits degenerate boundary conditions, consistent with whole class of symmetries, and therefore has a sequence of the reductions which are solved in elementary functions. The corresponding classes of reduced systems are studied in 1980-1990 by Ukrainian school, group leaded by Yu.M.Berezanskii.

Let's move on to the class of two-dimensional lattices

$$u_{n,xy} = f(u_{n+1}, u_n, u_{n-1}, u_{n,x}, u_{n,y}) \tag{1}$$

**Definition** We call equation (1) integrable if there are functions $f_0(u_1, u_0, u_{0,x}, u_{0,y})$ and $f_N(u_N, u_{N-1}, u_{N,x}, u_{N,y})$ such that system

$$\begin{aligned} u_{0,x,y} &= f_0(u_1, u_0, u_{0,x}, u_{0,y}) \\ u_{n,xy} &= f(u_{n+1}, u_n, u_{n-1}, u_{n,x}, u_{n,y}), \quad 1 \leq n \leq N, \\ u_{N,x,y} &= f_N(u_N, u_{N-1}, u_{N,x}, u_{N,y}) \end{aligned} \tag{2}$$

obtained from (1) is integrable in the sense of Darboux for any choice of the integer $N$.

Motivated by the example

$$\begin{aligned} u_{0,xy} &= e^{u_1 - 2u_0}, \\ u_{n,xy} &= e^{u_{n+1} - 2u_n + u_{n-1}}, \quad 1 \leq n \leq N, \\ u_{N,xy} &= e^{-2u_N + u_{N-1}} \end{aligned}$$

obtained from the Toda lattice is integrable in the sense of Darboux.

Let us show the list of integrable Toda type lattices given in [A.B. Shabat, R.I. Yamilov, To a transformation theory of two-dimensional integrable systems, Phys. Lett. A **227** (1997) 15-23]

1) $u_{n,xy} = e^{u_{n+1}-2u_n+u_{n-1}}$,
2) $u_{n,xy} = e^{u_{n+1}} - 2e^{u_n} + e^{u_{n-1}}$,
3) $u_{n,xy} = e^{u_{n+1}-u_n} - e^{u_n-u_{n-1}}$,
4) $u_{n,xy} = (u_{n+1} - 2u_n + u_{n-1})\, u_{n,x}$,
5) $u_{n,xy} = (e^{u_{n+1}-u_n} - e^{u_n-u_{n-1}})\, u_{n,x}$,
6) $u_{n,xy} = \alpha_n u_{n,x} u_{n,y}, \quad \alpha_n = \frac{1}{u_n-u_{n-1}} - \frac{1}{u_{n+1}-u_n} = \frac{u_{n+1}-2u_n+u_{n-1}}{(u_{n+1}-u_n)(u_n-u_{n-1})}$,

One more was found in [M. N. Poptsova, I. T. Habibullin, "Algebraic properties of quasilinear two-dimensional lattices connected with integrability", Ufa Math. J., 10:3 (2018), 86–105]

7) $u_{n,xy} = \alpha_n (u_{n,x} - u_n^2 - 1)(u_{n,y} - u_n^2 - 1) + 2u_n (u_{n,x} + u_{n,y} - u_n^2 - 1)$.

Recall that Darboux integrability means that system (2) possesses $N+1$ nontrivial integrals in both characteristic directions. The function $\bar{u} = (u_0, \ldots, u_{N0})$ and its derivatives $\bar{u}_x$, $\bar{u}_y$, $\bar{u}_{xx}$, $\bar{u}_{yy}$, etc., are taken as dynamical variables. By definition, a function $I(\bar{u}, \bar{u}_x, \bar{u}_{xx}, \ldots)$ depending on a finite set of dynamical variables is an $x$-integral of system (2) if $D_y I = 0$ where $D_y$ is the operator of total derivative with respect to the variable $y$. That is to say $I$ is found from the system

$$YI = 0, \quad X_i I = 0$$

where

$$X_i = \frac{\partial}{\partial u_{i,y}}, \quad Y = \sum_{i=0}^{N} \left( u_{i,y} \frac{\partial}{\partial u_i} + f_i \frac{\partial}{\partial u_{i,x}} + D_x(f_i) \frac{\partial}{\partial u_{i,xx}} + \cdots \right)$$

and $f_i = f(u_{i+1}, u_i, u_{i-1}, u_{i,x}, u_{i,y})$ for $i = 1, 2, \ldots N-1$.

Let us consider the Lie algebra $L_y$ generated by the operators $Y$, $X_i$, $i = 0, ...N$ over the ring $K$ of locally analytic functions of the dynamical variables $\bar{u}_y, \bar{u}, \bar{u}_x, \bar{u}_{xx}, \dots$. To the standard operation $[Z, W] = ZW - WZ$ we add the following conditions: for any $Z, W \in L_y$ and $a, b \in K$ we require

$$[Z, aW] = Z(a)W + a[Z, W],$$
$$(aZ)b = aZ(b).$$

These conditions mean that if $Z \in L_y$ and $a \in K$ then $aZ \in L_y$. The algebra $L_y$ defined in this way is called the Lie-Rinehart algebra. We call it characteristic algebra in $y$-direction. In a similar way characteristic algebra $L_x$ is defined.
The algebra $L_y$ is of finite dimension if it admits a finite basis of operators $Z_1, Z_2, \dots, Z_k \in L_y$ such that an arbitrary element $Z \in L_y$ can be represented as their linear combination:
$Z = a_1 Z_1 + a_2 Z_2 + \dots + a_k Z_k$; here the coefficients are functions $a_1, a_2, \dots, a_k \in K$.

The definition can be found in [G. Rinehart, Differential forms for general commutative algebras, Trans. Amer. Math. Soc. **108** (1963) 195-222.] We thank D Millionshchikov for drawing our attention to the correct name of this object.

Let us briefly discuss the difference between Lie and Lie-Rinehart algebras.

**Example 3.1.** Obviously, the Lie algebra generated by the operators $Z_1 = x^2\frac{\partial}{\partial x}$ and $Z_2 = x^3\frac{\partial}{\partial x}$ is infinite-dimensional. For example, the commutator $[Z_1, Z_2] = x^4\frac{\partial}{\partial x}$ is not a linear combination of $Z_1$ and $Z_2$ with constant coefficients. At the same time the Lie-Rinehart algebra corresponding to the ring $A$ of functions analytic in the domain $x \neq 0$ of the complex plane generated by the same operators is one-dimensional, since any element $Z$ in the algebra can be represented as $Z = f(x)Z_0$, where $Z_0 = \frac{\partial}{\partial x}$, since here linear combinations with variable coefficients are allowed.

Our approach is based on the following key statement [A.V. Zhiber, R.D. Murtazina, I.T. Habibullin, A.B. Shabat, Characteristic Lie rings and nonlinear integrable equations, M.-Izhevsk: Institute of Computer Science, (2012) 376 pp.] :

### Theorem

*System (2) admits a complete set of $y$-integrals ($x$-integrals) if and only if its characteristic algebra $L_y$ (respectively, $L_x$) is of finite dimension.*

### Corollary

*System (2) is integrable in the sense of Darboux if both characteristic algebras $L_x$ and $L_y$ are of finite dimension.*

By using the method of characteristic algebras it can be proved
**Proposition 1**. Integrable equation of the form

$$u_{n,xy} = s(u_{n+1}, u_n, u_{n-1})u_{n,x}u_{n,y} + \beta(u_{n+1}, u_n, u_{n-1})u_{n,x} + \\ + \gamma(u_{n+1}, u_n, u_{n-1})u_{n,y} + \delta(u_{n+1}, u_n, u_{n-1}),$$

where at least one of the conditions $\frac{\partial s(u_{n+1},u_n,u_{n-1})}{\partial u_{n\pm 1}} \neq 0$ holds, can be reduced by a point transformation to one of the following forms:

$$u_{n,xy} = \alpha_n u_{n,x}u_{n,y},$$
$$u_{n,xy} = \alpha_n(u_{n,x} - u_n^2 - 1)(u_{n,y} - u_n^2 - 1) + \\ +2u_n(u_{n,x} + u_{n,y} - u_n^2 - 1).$$
$$\alpha_n = \frac{1}{u_n - u_{n-1}} - \frac{1}{u_{n+1} - u_n} = \frac{u_{n+1} - 2u_n + u_{n-1}}{(u_{n+1} - u_n)(u_n - u_{n-1})},$$

[I. Habibullin, M. Poptsova (Kuznetsova), Algebraic properties of quasilinear two-dimensional lattices connected with integrability, Ufa Math. J. **10**, no. 3 (2018) 86-105.]

$$u_{n,xy} = \alpha_n u_{n,x} u_{n,y},$$

$$u_{n,xy} = \alpha_n (u_{n,x} - u_n^2 - 1)(u_{n,y} - u_n^2 - 1) +$$
$$+2u_n(u_{n,x} + u_{n,y} - u_n^2 - 1).$$

$$\alpha_n = \frac{1}{u_n - u_{n-1}} - \frac{1}{u_{n+1} - u_n} = \frac{u_{n+1} - 2u_n + u_{n-1}}{(u_{n+1} - u_n)(u_n - u_{n-1})},$$

The first equation has been found by E.V.Ferapontov [Theoret. and Math. Phys. **110** (1997) 68-77.]

and independently by A.B.Shabat and R.I.Yamilov [Phys. Lett. A **227** (1997) 15-23.]

The second was found in [I. Habibullin, M. Poptsova (Kuznetsova), Ufa Math. J. **10**, no. 3 (2018) 86-105.]

The Lax pair for the second one was found in M. N. Kuznetsova, “Lax Pair for a Novel Two-Dimensional Lattice”, SIGMA, 17 (2021),

In [M.N. Kuznetsova, Classification of a subclass of quasilinear two-dimensional lattices by means of characteristic algebras, Ufa Math. J. **11**, no. 3 (2019) 109-131.] the following statement is proved.
**Proposition 2**. Integrable equation of the form

$$u_{n,xy} = g(u_{n+1}, u_n, u_{n-1})u_{n,y} + \beta(u_{n+1}, u_n, u_{n-1})u_{n,x} + \\ +\delta(u_{n+1}, u_n, u_{n-1}), \tag{3}$$

where the coefficients satisfy at least one of the four conditions $\frac{\partial g}{\partial u_{n\pm1}} \neq 0$, $\frac{\partial \beta}{\partial u_{n\pm1}} \neq 0$, can be reduced by a point transformation to one of the following forms:

$$u_{n,xy} = \left(e^{u_n - u_{n-1}} - e^{u_{n+1} - u_n}\right) u_{n,y}, \tag{4}$$

$$u_{n,xy} = (u_{n+1} - 2u_n + u_{n-1})\, u_{n,y}. \tag{5}$$

Equations (4) and (5) were found earlier in [A.B. Shabat, R.I. Yamilov, To a transformation theory of two-dimensional integrable systems, Phys. Lett. A **227** (1997) 15-23.].

The following sub-case turned out to be very difficult.

**Proposition 3**. A lattice of the form

$$u_{n,xy} = g(u_{n+1}, u_n, u_{n-1}), \tag{6}$$

which is integrable in the sense of Definition 1, can be reduced by suitable rescalings to one of the following forms:

$$u_{n,xy} = e^{2u_n - mu_{n+1} - ku_{n-1}} + a(u_{n+1}, u_n) + b(u_n, u_{n-1}), \tag{7}$$

$$u_{n,xy} = e^{u_n} u_{n+1} u_{n-1} + a(u_{n+1}, u_n) + b(u_n, u_{n-1}), \tag{8}$$

$$u_{n,xy} = u_{n+1} u_{n-1} + a(u_{n+1}, u_n) + b(u_n, u_{n-1}), \tag{9}$$

$$u_{n,xy} = a(u_{n+1}, u_n) + b(u_n, u_{n-1}); \tag{10}$$

here $m$, $k$ are positive integers.

[I.T. Habibullin, M.N. Kuznetsova, A.U. Sakieva, Integrability conditions for two-dimensional lattices, J. Phys. A: Math. Theor. 53 (2020) 395203 (25pp)]

What we failed to do?
1) General case (is not linear with respect to $u_{n,x}$ and/or $u_{n,y}$ )

$$u_{n,xy} = f(u_{n+1}, u_n, u_{n-1}, u_{n,x}, u_{n,y}),$$

Need a new idea.
1) Toda type case

$$u_{n,xy} = g(u_{n+1}, u_n, u_{n-1}),$$

First reason: this case is too labor consuming. Second reason is related with Yamilov's plausible reasoning. This equation is integrable if and only if an equation $u_{n,xx} = g(u_{n+1}, u_n, u_{n-1})$ is integrable. There are only three integrable equations of that form

1) $u_{n,xx} = e^{u_{n+1}-2u_n+u_{n-1}}$,
2) $u_{n,xx} = e^{u_{n+1}} - 2e^{u_n} + e^{u_{n-1}}$,
3) $u_{n,xx} = e^{u_{n+1}-u_n} - e^{u_n-u_{n-1}}$.

# A combined method of classification

In the paper [E. V. Ferapontov, I. T. Habibullin, M. N. Kuznetsova, V. S. Novikov, “On a class of 2D integrable lattice equations”, Journal of Mathematical Physics, 61:7 (2020)] a new approach to the classification of integrable lattice type equations in 3D was developed by combining the geometric approach of [E.V. Ferapontov, B.S. Kruglikov, Dispersionless integrable systems in 3D and Einstein-Weyl geometry, J. Diff. Geom. **97** (2014) 215-254] with the test based on the requirement of Darboux integrability of suitably reduced equations. As an illustration we classify integrable equations of the form

$$u_{xy} = f(u, u_x, u_y, \triangle_z u \triangle_{\bar z} u, \triangle_{z\bar z} u). \tag{11}$$

Notations: $\triangle_z = \frac{T_z - 1}{\epsilon}$, $\triangle_{\bar z} = \frac{1 - T_{\bar z}}{\epsilon}$ for the forward/backward discrete derivatives and $\triangle_{z\bar z} = \frac{T_z + T_{\bar z} - 2}{\epsilon^2}$ for the symmetrised second-order discrete derivative; here $T_z, T_{\bar z}$ are the forward/backward $\epsilon$-shifts in the variable $z$.

Equivalent form of the equation

$$u_{n,xy} = g(u_n, u_{n,x}, u_{n,y}, (u_{n+1} - u_n)(u_n - u_{n-1}), u_{n+1} - 2u_n + u_{n-1}), \tag{12}$$

Familiar examples of type (11) include the Toda equation

$$u_{xy} = e^{\triangle_{z\bar{z}} u} \tag{13}$$

and the equation

$$u_{xy} = u_x u_y \frac{\triangle_{z\bar{z}} u}{\triangle_z u \triangle_{\bar{z}} u} \tag{14}$$

found by Ferapontov, Shabat and Yamilov. Note that dispersionless limits of the above equations (obtained as $\epsilon \to 0$) coincide with the Boyer-Finley equation $u_{xy} = e^{u_{zz}}$ and the equation $u_{xy} = \frac{u_x u_y}{u_z^2} u_{zz}$, respectively. Both limits belong to the class of dispersionless integrable PDEs.

The above examples suggest the following 2-step classification procedure:
(1) First we classify integrable equation of the form

$$u_{xy} = F(u, u_x, u_y, u_z, u_{zz}), \tag{15}$$

which can be viewed as dispersionless limits of equations (11) when $\epsilon \to 0$. This can be done by requiring that the characteristic conformal structure $[g]$ of equation (15), namely

$$[g] = 4F_{u_{zz}} dxdy - dz^2, \tag{16}$$

is Einstein-Weyl on every solution of (15).
(2) Secondly, replacing $u_z$ and $u_{zz}$ in the equations obtained at the previous step by $\sqrt{\triangle_z u \triangle_{\bar{z}} u}$ and $\triangle_{z\bar{z}} u$, respectively, we obtain equations of type (11) which, at this stage, are our candidates for integrability. To these candidate equations we apply the test of Darboux integrability of reductions obtained by imposing suitable cut-off conditions.

**Genuinely nonlinear case $F_{u_{zz}u_{zz}} \neq 0$.**

$$\beta'(u)u_{xy} + \beta''(u)u_x u_y = \gamma e^{\beta'(u)u_{zz} + \beta''(u)u_z^2} + \delta \tag{17}$$

and

$$\beta'(u)u_{xy} + \beta''(u)u_x u_y = \gamma e^{\beta'(u)u_{zz} + \beta''(u)u_z^2 + \delta\beta'(u)u_z + \frac{2}{9}\delta^2\beta(u)}; \tag{18}$$

here $\beta(u)$ is an arbitrary function and $\gamma, \delta$ are constants (without any loss of generality one can set $\gamma = 1$). Note that although $\beta(u)$ can be eliminated by a change of variables $\tilde{u} = \beta(u)$, this only works at the dispersionless level and is not necessarily valid for the corresponding lattice equations obtained by replacing $u_z$ and $u_{zz}$ with $\sqrt{\triangle_z u \triangle_{\bar{z}} u}$ and $\triangle_{z\bar{z}} u$. Thus, at this stage we will keep $\beta(u)$ arbitrary.
This case produces the lattice

$$u_{n,xy} = e^{\triangle_{z\bar{z}} u}.$$

# Quasilinear case $F_{u_{zz}u_{zz}} = 0$.

Let us set

$$u_{xy} = \varphi(u, u_x, u_y, u_z)u_{zz} + \psi(u, u_x, u_y, u_z).$$

**Subcase 1:** coefficient $\varphi$ depends on $u$ only, $\varphi_u \neq 0$. In this case the Einstein-Weyl conditions lead to the following integrable dispersionless equation:

$$u_{xy} = \beta u_{zz} + \frac{3}{2}\alpha\beta u_z + \frac{\alpha^2\beta^2}{2\beta'} + \left(\frac{\beta'}{\beta} - \frac{\beta''}{\beta'}\right)u_x u_y + \frac{\beta\beta''}{\beta'}u_z^2. \quad (19)$$

**Subcase 2:** coefficient $\varphi$ depends on $u, u_z$ only, $\varphi_{u_z} \neq 0$. In this case we have three integrable dispersionless equations:

$$u_{xy} = \gamma e^{\beta u_z} \left( u_{zz} + \frac{\beta'}{\beta} u_z^2 \right) + \frac{\delta}{\beta} - \frac{\beta'}{\beta} u_x u_y, \tag{20}$$

$$u_{xy} = e^{\alpha\beta + \beta' u_z} \left( u_{zz} + \alpha u_z + \frac{\alpha}{2\beta'} + \frac{\beta''}{\beta'} u_z^2 \right) - \frac{\beta''}{\beta'} u_x u_y, \tag{21}$$

$$u_{xy} = e^{\frac{1}{2}\alpha\beta + \beta' u_z} \left( u_{zz} + \frac{1}{2}\alpha u_z + \frac{\alpha}{\beta'} + \frac{\beta''}{\beta'} u_z^2 \right) - \frac{\beta''}{\beta'} u_x u_y. \tag{22}$$

These cases lead to the lattice

$$u_{xy} = \frac{u_x u_y}{u} + u \triangle_{z\bar{z}} u \quad \text{or} \quad v_{n,xy} = e^{v_{n+1}} - 2e^{v_n} + e^{v_{n-1}}, u = e^v$$

**Subcase 3:** coefficient $\varphi$ depends on $u, u_z, u_y$ only, $\varphi_{u_y} \neq 0$. In this case we have four integrable dispersionless equations:

$$u_{xy} = \beta' u_y u_{zz} + \left(\frac{1}{2}\alpha^2\beta + \frac{3}{2}\alpha\beta' u_z + \beta'' u_z^2\right) u_y - \frac{\beta''}{\beta'} u_x u_y, \tag{23}$$

$$u_{xy} = (\gamma + \beta u_y)\left(u_{zz} + \frac{\delta}{\beta} + \frac{\beta'}{\beta} u_z^2\right) - \frac{\beta'}{\beta} u_x u_y, \tag{24}$$

$$u_{xy} = \gamma e^{\frac{1}{2}\alpha\beta + \beta' u_z} u_y(\alpha + 2\beta' u_{zz} + \alpha\beta' u_z + 2\beta'' u_z^2) - \frac{\beta''}{\beta'} u_x u_y, \tag{25}$$

$$u_{xy} = \delta e^{\beta u_z}\left(u_y + \frac{\gamma}{\beta}\right)(\beta u_{zz} + \beta' u_z^2) - \frac{\beta'}{\beta} u_x u_y. \tag{26}$$

This case produces the lattice

$$u_{xy} = u_x \triangle_{z\bar{z}} u. \tag{27}$$

**Subcase 4:** coefficient $\varphi$ depends on all four arguments $u, u_z, u_y, u_x$, we can assume $\varphi_{u_x} \neq 0,\ \varphi_{u_y} \neq 0$. In this case we have the following equations:

$$u_{xy} = \frac{2u_{zz} + (4\beta' - \alpha)u_z + 2\beta\beta' - \alpha\beta}{2(u_z + \beta)^2} u_x u_y, \tag{28}$$

$$u_{xy} = \frac{u_x u_y + \beta u_x}{(u_z + \gamma\beta)^2} u_{zz} + \frac{(4\gamma\beta' - \alpha)u_z + 2\gamma^2\beta\beta' - \alpha\gamma\beta}{2(u_z + \gamma\beta)^2} u_x u_y - \\ - \frac{2\beta' u_z^2 + \alpha\beta u_z + \alpha\gamma\beta^2}{2(u_z + \gamma\beta)^2} u_x, \tag{29}$$

$$
\begin{aligned}
u_{xy} = {} & \frac{(u_x+\beta)(u_y+\delta\beta)}{(u_z+\gamma\beta)^2}u_{zz} + \\
& + \frac{(4\gamma\beta'-\alpha)u_z + 2\gamma^2\beta\beta' - \alpha\gamma\beta}{2(u_z+\gamma\beta)^2}u_x u_y - \\
& \quad - \frac{2\beta' u_z^2 + \alpha\beta u_z + \alpha\gamma\beta^2}{2(u_z+\gamma\beta)^2}(u_y + \delta u_x + \delta\beta),
\end{aligned} \tag{30}
$$

From these three equations above we find

$$
\begin{aligned}
& u_{n,xy} = \alpha_n u_{n,x} u_{n,y}, \\
& u_{n,xy} = \alpha_n (u_{n,x} - u_n^2 - 1)(u_{n,y} - u_n^2 - 1) + \\
& \quad +2u_n(u_{n,x} + u_{n,y} - u_n^2 - 1). \\
& \alpha_n = \frac{1}{u_n - u_{n-1}} - \frac{1}{u_{n+1} - u_n} = \frac{u_{n+1} - 2u_n + u_{n-1}}{(u_{n+1} - u_n)(u_n - u_{n-1})},
\end{aligned}
$$

We also have the following three equations involving hyperbolic functions:

$$u_{xy} = \beta' \frac{\beta' u_{zz} + \frac{1}{2}\alpha\beta' u_z + \beta'' u_z^2}{\sinh^2(\gamma + \frac{1}{2}\alpha\beta + \beta' u_z)} u_x u_y - \frac{\beta''}{\beta'} u_x u_y, \tag{31}$$

$$u_{xy} = \frac{\beta u_{zz} + \beta' u_z^2}{\sinh^2(\delta + \beta u_z)} u_x(\gamma + \beta u_y) - \frac{\beta'}{\beta} u_x u_y, \tag{32}$$

$$\begin{aligned} u_{xy} = {} & \frac{(\mu + \beta u_x)(\nu + \beta u_y)}{\sinh^2(\delta + \beta u_z)} u_{zz} + \\ & + \frac{\beta'}{\beta} \frac{\mu\nu + \beta(\mu u_y + \nu u_x + \beta u_x u_y)}{\sinh^2(\delta + \beta u_z)} u_z^2 - \frac{\beta'}{\beta} u_x u_y. \end{aligned} \tag{33}$$

They do not produce integrable lattices.

Comments to the second classification algorithm:
1) All found equations are quasilinear (linear on $u_{n,x}$, $u_{n,x}$)
2) All equations known before are in the list, except those that are not represented in the form

$$u_{n,xy} = g(u_n, u_{n,x}, u_{n,y}, (u_{n+1} - u_n)(u_n - u_{n-1}), u_{n+1} - 2u_n + u_{n-1}),$$

3) $u_{n,xy} = e^{u_{n+1}-u_n} - e^{u_n-u_{n-1}},$

5) $u_{n,xy} = \left(e^{u_{n+1}-u_n} - e^{u_n-u_{n-1}}\right) u_{n,x},$

Integrable equations of the form

$$u^j_{n+1,x} = F(u^j_{n,x}, u^{j+1}_n, u^j_{n+1}, u^j_n, u^{j-1}_{n+1}), \quad -\infty < n, j < \infty, \tag{34}$$

have been studied by many authors (see Ferapontov, E.V., V.S. Novikov, and I. Roustemoglou. "On the classification of discrete Hirota-type equations in 3D." Int. Math. Res. Not. IMRN 2015 and the references therein).

# Conjecture

A lattice of the form

$$u^j_{n+1,x} = F(u^j_{n,x}, u^{j+1}_n, u^j_{n+1}, u^j_n, u^{j-1}_{n+1}), \quad -\infty < n, j < \infty, \tag{35}$$

is integrable if and only if there exists a pair of functions $H^{(1)}$ and $H^{(2)}$ such that for any choice of the integer $N$, a system of the hyperbolic type differential-difference equations

$$\begin{aligned} u^1_{n+1,x} &= H^{(1)}(u^1_{n,x}, u^2_n, u^1_{n+1}, u^1_n), \\ u^j_{n+1,x} &= F^j(u^j_{n,x}, u^{j+1}_n, u^j_{n+1}, u^j_n, u^{j-1}_{n+1}), \quad 1 < j < N, \\ u^N_{n+1,x} &= H^{(2)}(u^N_{n,x}, u^N_{n+1}, u^N_n, u^{N-1}_{n+1}), \quad -\infty < n < \infty \end{aligned} \tag{36}$$

obtained from (35) is integrable in the sense of Darboux. We have checked for $N \le 3$ that known integrable lattices pass this test.

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