On some linear equations connected with dispersionless integrable systems

L.V. Bogdanov

L.D. Landau ITP RAS

15-12-2021

L V Bogdanov, SDYM equations on the self-dual background, Journal of Physics A (Letter), 50(19) 19LT02 (2017)
L.V. Bogdanov, Matrix extension of the Manakov-Santini system and an integrable chiral model on an Einstein-Weyl background, Theor. Math. Phys., 201(3), 1701-1709 (2019)
L V Bogdanov, Dispersionless integrable systems and the Bogomolny equations on an Einstein-Weyl geometry background, Theor. Math. Phys., 205(1), 1280-1291 (2020)
L.V. Bogdanov, Matrix extension of multidimensional dispersionless integrable hierarchies, Theoret. Math. Phys., 209(1), 1319-1329 (2021)

Matrix extension of multidimensional dispersionless integrable systems

Multidimensional dispersionless integrable systems: Lax pairs of the type

$$[X_1, X_2] = 0,$$

$$X_1 = \partial_{t_1} + \sum_{i=1}^{N} F_i \partial_{x_i} + F_0 \partial_\lambda, \quad X_2 = \partial_{t_2} + \sum_{i=1}^{N} G_i \partial_{x_i} + G_0 \partial_\lambda.$$

λ - 'spectral parameter', functions F_k, G_k are holomorphic in λ and depend on the variables t_1, t_2, x_n. We will consider polynomials (or Laurent polynomials) in λ (meromorphic functions).

Dispersionless limits of integrable equations (dKP, dispersionless 2DTL hierarchy), Plebański heavenly equations, hyper-Kähler hierarchies belong to this class. Twistor integrability.

Higher flows, the hierarchy, dressing scheme based on nonlinear vector Riemann-Hilbert problem

$$\Psi_+ = \mathrm{F}(\Psi_-), \quad \Psi = (\psi_0, \ldots, \psi_N)$$

Matric extension - extended (gauge covariant) vector fields of the form

$$\nabla_{X_1} = X_1 + A_1, \quad \nabla_{X_2} = X_2 + A_2,$$

A_1, A_2 are matrix functions of space-time variables holomorphic in λ (polynomials, Laurent polynomials, meromorphic functions).
Lax pairs of this structure were already present in Zakharov and Shabat (1979).
The commutator of two covariant vector fields contains vector field part and matrix (Lie algebraic) part,

$$[\nabla_{X_1}, \nabla_{X_2}] = [X_1, X_2] + X_1 A_2 - X_2 A_1 + [A_1, A_2]$$

Compatibility condition - vector fields (dispersionless equations)

$$[X_1, X_2] = 0$$

Compatibility condition - matrix part (matrix equations on the dispersionless background)

$$X_1 A_2 - X_2 A_1 + [A_1, A_2] = 0$$

Matrix dressing on the background

Wave function for extended system

$$\nabla_{X_1}\Phi = (X_1 + A_1)\Phi = 0, \quad \nabla_{X_2}\Phi_0 = (X_2 + A_2)\Phi = 0,$$

can be found through the Riemann-Hilbert problem

$$\Phi_+ = \Phi_- R(\psi_0, \dots, \psi_N),$$

defined on some oriented curve γ in the complex plane, where ψ_i are wave functions of linear operators of basic dispersionless system

$$X_1\psi_i = 0, \quad X_2\psi_i = 0.$$

The hierarchy can also be considered.

Local form of SD conformal structure

Theorem (Dunajski, Ferapontov and Kruglikov (2014))

There exist local coordinates (z, w, x, y) such that any ASD conformal structure in signature $(2, 2)$ is locally represented by a metric

$$\frac{1}{2}g = dwdx - dzdy - F_y dw^2 - (F_x - G_y)dwdz + G_x dz^2,$$

where the functions $F, \ G : M^4 \to \mathbb{R}$ satisfy a coupled system of third-order PDEs,

$$\begin{aligned} &\partial_x(Q(F)) + \partial_y(Q(G)) = 0, \\ &(\partial_w + F_y\partial_x + G_y\partial_y)Q(G) + (\partial_z + F_x\partial_x + G_x\partial_y)Q(F) = 0, \end{aligned} \tag{1}$$

where

$$Q = \partial_w\partial_x - \partial_z\partial_y + F_y{\partial_x}^2 - G_x{\partial_y}^2 - (F_x - G_y)\partial_x\partial_y.$$

System (1) arises as $[X_1, X_2] = 0$ from the dispersionless Lax pair

$$X_1 = \partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_x\partial_\lambda,$$
$$X_2 = \partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_y\partial_\lambda.$$

Due to compatibility conditions, f_1 and f_2 can be expressed through F and G,

$$f_x = -Q(G), \quad f_y = Q(F),$$
$$Q = \partial_w\partial_x - \partial_z\partial_y + F_y{\partial_x}^2 - G_x{\partial_y}^2 - (F_x - G_y)\partial_x\partial_y.$$

Correspondence between ASD conformal structures and integrable system defined by generic commuting vector fields.
Real case with the signature (2,2) or, generally, complex analytic case may be considered.
Reductions:
Dunajski system - null Kähler case, divergence free vector fields
$f_1, f_2 = 0$ (no ∂_λ in the vector fields), divergence free - *Plebanski's second heavenly equation* (ASD, Ricci flat)

Extension of the Lax pair

Consider a gauge field A in some (matrix) Lie algebra and 'covariant vector fields' X_1, X_2

$$\nabla_{X_1} = \partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_1\partial_\lambda + A_1,$$
$$\nabla_{X_2} = \partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_2\partial_\lambda + A_2$$

(here A_1, A_2 do not depend on λ).
Commutation relation give the system describing conformally ASD metric and the system for A_1, A_2

$$\partial_x A_2 = \partial_y A_1,$$
$$(\partial_z + F_x\partial_x + G_x\partial_y)A_2 - (\partial_w + F_y\partial_x + G_y\partial_y)A_1 + [A_1, A_2] = 0,$$

or, in terms of matrix potential Φ_1, $A_1 = \partial_x\Phi_1$, $A_2 = \partial_y\Phi_1$,

$$Q\Phi_1 = [\Phi_{1x}, \Phi_{1y}],$$
$$Q = \partial_w\partial_x - \partial_z\partial_y + F_y{\partial_x}^2 - G_x{\partial_y}^2 - (F_x - G_y)\partial_x\partial_y$$

SDYM equations on conformal structure background

SDYM (ASDYM) equations

$$\mathsf{F} = \pm * \mathsf{F}$$

represent SD (ASD) condition for the two-form (field intensity, connection curvature)

$$\mathsf{F} = \mathrm{d}\mathsf{A} + \mathsf{A} \wedge \mathsf{A}$$

the gauge field (potential) A is a one-form (connection form) taking its values in some Lie algebra (we will consider general matrix-valued form). Full Yang-Mills equation

$$D * \mathsf{F} = 0$$

SDYM are conformally invariant and depend only on conformal structure. SD (ASD) conformal structure $[g]$ (ASD or SD part of the Weyl tensor vanishes):

$$W = \pm * W,$$

for real case exists only for Riemannian (Euclidean) signature and neutral signature $(+ + - -)$. Complexification allows to consider both cases on equal footing.

Integrable background geometries

Atiyah M.F., Hitchin N.J., Singer I.M., Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. London Ser. A 362 (1978), 425–461.

There is a curved twistor space as long as the conformal structure on 4-manifold is selfdual. SDYM equations for selfdual conformal structure are integrable by twistor approach.

David M.J. Calderbank, SIGMA 10 (2014), 034, 51 pages

SDYM equations (and their reductions) are integrable in some nonflat geometries described by dispersionless integrable equations.

We will go opposite direction, starting from dispersionless integrable equations and extending integrable structures (Lax pairs, dressing scheme, the hierarchy) for gauge field equations on the background.

SDYM equations on integrable background and an extended Lax pair

1. Geometry

The system engendered by the extended Lax pair gives a general form of SDYM equations for arbitrary SD conformal structure in signature (2,2) (locally, up to transformations of coordinates and a gauge).

2. Integrability

Extended Lax pair belongs to the hierarchy which intertwines SDYM hierarchy and 4-dimensional dispersionless hierarchy. The Lax-Sato equations and dressing scheme can be constructed for this hierarchy.

Extension – Abelian case

Scalar extension of the Lax pair

$$
\begin{aligned}
\nabla_{X_1} &= \partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_1\partial_\lambda + a_1,\\
\nabla_{X_2} &= \partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_2\partial_\lambda + a_2
\end{aligned}
$$

Linear equation for the potential ϕ_1, $a_1 = \partial_x\phi_1$, $a_2 = \partial_y\phi_1$

$$
Q\phi_1 = 0, \quad Q = \partial_w\partial_x - \partial_z\partial_y + F_y{\partial_x}^2 - G_x{\partial_y}^2 - (F_x - G_y)\partial_x\partial_y
$$

1. Self-dual electromagnetic field on the background of the self-dual conformal structure.
2. General solution given by explicit formula
3. Connected with the linearisation of the basic (geometric) equations

Solution

Scalar Riemann-Hilbert problem can be solved explicitly:

$$\phi_{\text{in}} = \phi_{\text{out}} R(\Psi^0, \Psi^1, \Psi^2),$$

$$\ln \phi_{\text{in}} - \ln \phi_{\text{out}} = r(\Psi^0, \Psi^1, \Psi^2), \quad R = \mathrm{e}^r,$$

$$\ln \phi = \frac{1}{2\pi i} \oint \frac{r(\Psi^0, \Psi^1, \Psi^2)}{\mu - \lambda} d\mu, \quad \phi = \exp\left(\frac{1}{2\pi i} \oint \frac{r(\Psi^0, \Psi^1, \Psi^2)}{\mu - \lambda} d\mu\right),$$

$$\phi_1 = -\frac{1}{2\pi i} \oint r(\Psi^0, \Psi^1, \Psi^2) d\mu \qquad (2)$$

Here $r(\Psi^0, \Psi^1, \Psi^2)$ is an arbitrary complex analytic function of its arguments. Any wave function $\Psi(\mathrm{t}, \lambda)$ defined on the contour gives a solution to equation of Abelian extension,

$$\phi_1 = -\frac{1}{2\pi i} \oint \Psi(\mathrm{t}, \mu) d\mu, \qquad (3)$$

It is easy to check this formula directly using the dispersionles Lax pair. Indeed,

$$(\partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_x\partial_\lambda)\Psi = 0,$$
$$(\partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_y\partial_\lambda)\Psi = 0,$$

implies

$$((\partial_w + F_y\partial_x + G_y\partial_y)\partial_x - (\partial_z + F_x\partial_x + G_x\partial_y)\partial_y)\,\Psi = (f_x\partial_y - f_y\partial_x)\partial_\lambda\Psi,$$

integration of the l.h.s. with respect to λ over a closed contour gives zero, thus $Q\oint\Psi d\lambda = 0$. For trivial background

$$(\partial_w\partial_x - \partial_z\partial_y)\,\phi_1 = 0,$$

$$\phi_1 = -\frac{1}{2\pi i}\oint r(\lambda, \lambda z + x, \lambda w + y)d\lambda.$$

This formula is easily recognised as a version of Penrose formula for solutions of the wave equation written for the case of neutral signature.

Linearisation

Using the standard basic set Ψ^0, Ψ^1, Ψ^2, we get

$$Qf = 0, \quad Q(F + zf) = 0, \quad Q(G + wf) = 0,$$

represents another form of conformal self-duality equations. Operator Q gives a simbol of linearization of these equations (highest order derivatives). Short-wave stability?
The second heavenly equation ($\Psi^0 = \lambda$, $f = 0$, divergence free vector fields)

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = 0,$$
$$Q = \partial_w\partial_x + \partial_z\partial_y + \Theta_{yy}\partial_x\partial_x + \Theta_{xx}\partial_y\partial_y - 2\Theta_{xy}\partial_x\partial_y,$$

Operator Q coincides with the linearisation. The metric (a general local form of self-dual Einstein metric (self-dual curvature tensor))

$$g = dwdx + dzdy - \Theta_{yy}dw^2 + 2\Theta_{xy}dwdz - \Theta_{xx}dz^2.$$

Solution to linearised equation $Q\phi_1 = 0$

$$\phi_1 = \frac{1}{2\pi i} \oint r(\mu, \Psi^1, \Psi^2) d\mu,$$

Ψ^1, Ψ^2 are basic wave functions of linear operators

$$(\partial_z - \lambda\partial_x + \Theta_{xy}\partial_x - \Theta_{xx}\partial_y)\Psi = 0,$$
$$(\partial_w - \lambda\partial_y + \Theta_{yy}\partial_x - \Theta_{xy}\partial_y)\Psi = 0.$$

In fact, any wave function $\Psi(\lambda)$ is a solution of linear equation $Q\Psi = 0$!
Implies the recursion for linear operator (similar to A Sergyeyev (2017)).

$$\partial_x\widetilde{\phi} = (\partial_z + \Theta_{xy}\partial_x - \Theta_{xx}\partial_y)\phi,$$
$$\partial_y\widetilde{\phi} = (\partial_w + \Theta_{yy}\partial_x - \Theta_{xy}\partial_y)\phi.$$

The dKP equation case

A simple (2+1)-dimensional example - the dKP equation

$$u_{xt} = u_{yy} + (uu_x)_x$$

Operator Q

$$Q = \partial_t\partial_x - \partial_y\partial_y - u\partial_{xx} - u_x\partial_x$$

Linearisation operator

$$P = \partial_t\partial_x - \partial_y\partial_y - u\partial_{xx} - 2u_x\partial_x - u_{xx}$$

Identity $\partial_x Q = P\partial_x$ implies that $Q\phi = 0 \Rightarrow P\partial_x\phi = 0$.

$$\phi = \frac{1}{2\pi i}\oint r(\Psi^0, \Psi^1)d\mu,$$

in standard dKP notations $\Psi^0 = L$, $\Psi^1 = M$.

Mikhalev-Pavlov equation

$$v_{xt} = v_{yy} + v_x v_{xy} - v_{xx} v_y$$

$$X_1 = \partial_y - \lambda\partial_x + v_x\partial_x,$$
$$X_2 = \partial_t - (\lambda^2 - v_x\lambda - v_y)\partial_x = \partial_t - \lambda\partial_y + v_y\partial_x$$

Non-Abelian extension

$$K_{tx} - K_{yy} + v_y K_{xx} - v_x K_{xy} = [K_x, K_y].$$

Abelian extension operator

$$Q = \partial_t\partial_x - \partial_y\partial_y + v_y\partial_x\partial_x - v_x\partial_x\partial_y, \qquad Q\Psi = 0.$$

Linearisation operator

$$P = \partial_t\partial_x - \partial_y\partial_y + v_y\partial_x\partial_x - v_x\partial_x\partial_y + v_{xx}\partial_y - v_{xy}\partial_x$$

Solution $P\Psi_x^{-1} = 0$, Ψ is a wave function of the Lax pair, recursion (A. Sergyeyev (2017))

Adjoint Lax operators and modifications

$$X_1 = \partial_{t_1} + \sum_{i=1}^{N} F_i \partial_{x_i} + F_0 \partial_\lambda, \quad X_2 = \partial_{t_2} + \sum_{i=1}^{N} G_i \partial_{x_i} + G_0 \partial_\lambda.$$

A basic set of wave functions $\psi_0, \ldots, \psi_N$, general wave function $\psi = F(\psi_0, \ldots, \psi_N)$, $X_1 \psi = 0$, $X_2 \psi = 0$.

$$-X^* = X + \text{div} X$$

A special solution $X^* J_0 = 0$, $J_0 = \frac{\partial(\psi_0, \ldots, \psi_N)}{\partial(\lambda, x_1, \ldots, x_N)}$

$$X \ln J + \text{div} X = 0, \quad \ln J = J_0 + F(\psi_0, \ldots, \psi_N)$$

$$X \ln(J^\alpha) + \alpha \text{div} X = 0,$$
$$(X + \alpha \text{div} X)(J^\alpha) = 0$$

Using these linear problems, we can add linear term to the operator Q, preserving solvability. For conformal self-duality equations,

$$Q' = Q + \alpha((F_{xx} + G_{xy})\partial_y - (F_{xy} + G_{yy})\partial_x)$$

Manakov-Santini system

The Manakov-Santini system – two-component integrable generalisation of the dispersionless Kadomtsev-Petviashvili equation,

$$u_{xt} = u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y,$$
$$v_{xt} = v_{yy} + u v_{xx} + v_x v_{xy} - v_{xx} v_y$$

Lax pair

$$X_1 = \partial_y - (\lambda - v_x)\partial_x + u_x \partial_\lambda,$$
$$X_2 = \partial_t - ((\lambda^2 - v_x\lambda + u - v_y)\partial_x + (u_x\lambda + u_y)\partial_\lambda$$

For $v = 0$ reduces to dKP (Khohlov-Zabolotskaya equation)

$$u_{xt} = u_{yy} + (uu_x)_x,$$

reduction $u = 0$ gives the equation (Mikhalev, Pavlov)

$$v_{xt} = v_{yy} + v_x v_{xy} - v_{xx} v_y.$$

Local form of EW geometry and dispersionless integrable systems

Theorem (Dunajski, Ferapontov and Kruglikov (2014))

There exists a local coordinate system (x, y, t) *on* M^3 *such that any Lorentzian Einstein-Weyl structure is locally of the form*

$$
\begin{aligned}
g &= -(dy + v_x dt)^2 + 4(dx + (u - v_y)dt)dt, \\
\omega &= v_{xx} dy + (-4u_x + 2v_{xy} + v_x v_{xx})dt,
\end{aligned}
$$

where the functions u *and* v *satisfy the Manakov-Santini system*

$$
\begin{aligned}
u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y, \\
v_{xt} &= v_{yy} + uv_{xx} + v_x v_{xy} - v_{xx} v_y
\end{aligned}
$$

Also valid for general complex analytic EW structure.

The Manakov-Santini system - extension

Non-Abelian extension

$$K_{tx} - K_{yy} - [K_x, K_y] - \partial_x(uK_x) + v_y K_{xx} - v_x K_{xy} = 0,$$

Tensor form

$$D\Phi + \tfrac{1}{2}\Phi\omega = *F,$$

This equation for Minkowski metric coincides the Yang-Mills-Higgs system introduced by Ward, leading to integrable chiral model. The term $\frac{1}{2}\omega\Phi$ is responsible for correct behavior under conformal gauge transformation $g \to fg$, $\Phi \to f^{-\frac{1}{2}}\Phi$.
Linear operator of Abelian extension

$$Q = \partial_t\partial_x - \partial_y\partial_y - u\partial_x\partial_x - u_x\partial_x + v_y\partial_x\partial_x - v_x\partial_x\partial_y$$

THANK YOU!