Symmetries of the second Painlevé equation

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Historical remarks & Outline

$$P_2[y(t);\alpha]: \qquad y'' = 2y^3 + ty + \alpha, \qquad y(t),\ t,\ \alpha \in \mathbb{C}.$$

Historical remarks

- For any $\alpha \in \mathbb{C}$ the Painlevé-2 equation defines new transcendental functions. [Golubev, 1950]
- When $\alpha \in \mathbb{Z}$, there exist rational solutions. [Yablonskii, 1959, Vorobiev, 1965]
- If $\alpha \in \mathbb{Z} + \frac{1}{2}$, the P_2 equation is reduced to the Airy equation. [Ablowitz and Segur, 1977]
- For $\alpha \notin \mathbb{Z}$ and $\alpha \notin \mathbb{Z} + \frac{1}{2}$, the Painlevé-2 equation defines only new transcendental functions. [Umemura and Watanabe, 1997]
- Determinant structures (in the Jacoby-Trudy form and in the Hankel form) of rational solutions. [Kajiwara and Ohta, 1996]
- The Hankel determinant structure of the (generic!) solution of the Toda lattice equation and its application to the Painlevé equations. [Kajiwara et al., 1999]
- Explanation of the Hankel determinant structure of the P_2 solutions. [Joshi et al., 2004]

Outline

- The second Painlevé equation and its Hamiltonian structure.
- Bäcklund transformations and their group structure.
- The symmetric form for the second Painlevé equation.
- τ-functions and τ-functions on the lattice.
- The Hankel determinant structure of the P_2 solutions.

Remark. We will follow the book [Noumi, 2004].

Painlevé-2 equation: a brief review

- The second Painlevé equation:

$$y'' = 2y^3 + ty + \left(b - \tfrac{1}{2}\right), \qquad y(z),\ z,\ b \in \mathbb{C}. \qquad \mathrm{P}_2$$

- Okamoto canonical coordinates:

$$q = y, \qquad p = y' + y^2 + \tfrac{1}{2} t.$$

- The Hamiltonian structure:

$$H = \tfrac{1}{2} p \left(p - 2q^2 - t\right) - b\, q, \qquad \{q, p\} = 1, \qquad \{q, q\} = \{p, p\} = 0.$$

- Motion equations:

$$\begin{cases} q' = -q^2 + p - \tfrac{1}{2} t, \\ p' = 2qp + b, \end{cases} \quad \Leftrightarrow \quad \mathrm{P}_2[q(z); b]. \qquad (1)$$

- The isomonodromic representation: $A_t - B_\lambda = [B, A] \quad \Leftrightarrow \quad (1)$,

$$B(t, \lambda) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} -q & 0 \\ -1 & q \end{pmatrix},$$

$$A(t, \lambda) = 2\lambda B + \begin{pmatrix} 0 & 2q^2 - p + t \\ 0 & 0 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} b & 0 \\ -2p & -b \end{pmatrix} \lambda^{-1}.$$

Painlevé-2 system: Bäcklund transformations

Definition. *A Bäcklund transformation* transforms one PDE solution to another its solution. *An auto-Bäcklund transformation* leaves a PDE invariant.

s-action

- $s\left(q,p;b\right)=\left(\tilde{q},\tilde{p};\tilde{b}\right)=\left(q+\frac{b}{p},p;-b\right)$:

$$\begin{aligned}\tilde{q}' &= q'-\frac{b}{p^2}p' \\ &= -q^2+p-\tfrac{1}{2}t-\frac{b}{p^2}\left(2qp+b\right) \\ &= -q^2+p-\tfrac{1}{2}t-2\frac{b}{p}q-\frac{b^2}{p^2} \\ &= -\tilde{q}^2+\tilde{p}-\tfrac{1}{2}t;\end{aligned}$$

$$\begin{aligned}\tilde{p}' &= p' \\ &= 2qp+b+b-b \\ &= 2\left(q+\frac{b}{p}\right)p-b \\ &= 2\tilde{q}\tilde{p}+\tilde{b}.\end{aligned}$$

- $s(y)=y+\dfrac{b}{y'+y^2+\frac{1}{2}t}$.

r-action

- $r\left(q,p;b\right)=\left(\hat{q},\hat{p};\hat{b}\right)=\left(-q,-p+2q^2+t;1-b\right)$.
- $r(y)=-y$.

Remark. s- and r-actions preserve the canonicity of coordinates q and p.

Bäcklund transformations: a group structure (1)

$$s\,(q,p;b) = \left(q + b\,p^{-1}, p; -b\right), \qquad r\,(q,p;b) = \left(-q, -p + 2q^2 + t; 1 - b\right).$$

Remark. Properties of a Bäcklund transformation **s**:

- $s(\varphi \pm \psi) = s(\varphi) \pm s(\psi), \qquad s(\varphi \cdot \psi) = s(\varphi) \cdot s(\psi), \qquad s(\varphi / \psi) = s(\varphi)/s(\psi);$
- $s(\varphi') = s(\varphi)'$, where $\varphi = \varphi(q(t), p(t), t)$.

- Fundamental relations: $s^2 = r^2 = 1$.

$$s^2\,(b) = s\,(-b) = b, \qquad r^2\,(b) = r\,(1-b) = 1 - r\,(b) = b,$$

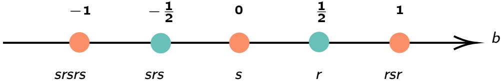
$$s^2\,(p) = p, \qquad r^2\,(q) = r\,(-q) = q,$$

$$s^2\,(q) = s\left(q + \frac{b}{p}\right) = s\,(q) + \frac{s\,(b)}{s\,(p)} = q; \qquad r^2\,(p) = r\left(-p + 2q^2 + t\right) = -r\,(p) + 2r\,(q)^2 + t = p.$$

- Compositions $T_n = (rs)^n$, $S_n = (rs)^n s$, $n \in \mathbb{N}$.

n	$\dots$	-2	-1	0	1	2	$\dots$
T_n	$\dots$	$srsr$	sr	1	rs	$rsrs$	$\dots$
S_n	$\dots$	$srsrs$	srs	s	r	rsr	$\dots$

- $T_n(b) = b - n, \qquad S_n(b) = n - b.$



Bäcklund transformations: a group structure (2)

Compositions $T_n = (rs)^n$ and $S_n = (rs)^n s$ of Bäcklund transformations

$$s\,(q, p; b) = \left(q + b\,p^{-1}, p; -b\right), \qquad r\,(q, p; b) = \left(-q, -p + 2q^2 + t; 1 - b\right)$$

form an affine Weyl group of type $A_1^{(1)}$

$$W\left(A_1^{(1)}\right) = \langle s, r\rangle = \{T_n,\ S_n,\ n \in \mathbb{N}\}, \qquad s^2 = r^2 = 1.$$

- Set $s_0 = rsr$, $s_1 = s$, $\pi = r$. Then an extension of the $W\left(A_1^{(1)}\right)$ group is

$$\tilde{W}\left(A_1^{(1)}\right) = \langle s_0, s_1; \pi\rangle, \qquad s_0^2 = s_1^2 = \pi^2 = 1, \qquad \pi s_i = s_{i+1}\pi, \qquad i \in \mathbb{Z}/2\mathbb{Z}.$$

Remark. Introduce new coordinates: $\sigma(t) = H\,(q(t), p(t), t) = \frac{1}{2}p(p - 2q^2 - t) - b\,q$. Then

$$p = -2\sigma', \qquad q = \frac{2\sigma'' + b}{4\sigma'},$$

and the so-called sigma form for the P_2 equation is

$$\left(\sigma''\right)^2 + 2\left(\sigma'\right)^3 + t\left(\sigma'\right)^2 - 2\sigma\sigma' = \frac{1}{4}b^2. \tag{2}$$

- Since $s(b^2) = b^2$, the rhs of (2) is invariant under the s-action and, therefore, $s(\sigma) = \sigma$.
- Regarding r-action, $r(\sigma) = r\,(H(b)) = H\,(b-1) = H\,(b) + q = \sigma + q$.
- But sigma-coordinates are not canonical: $\Omega = d\sigma'' \wedge d\,(\ln \sigma')$.

Symmetric form (1)

- Coordinates for the so-called symmetric form:

$$f_1 = p, \qquad \alpha_1 = b,$$
$$f_0 = \pi(p) = -p + 2q^2 + t, \qquad \alpha_0 = \pi(b) = 1 - b.$$

- Then the P_2 equation is equivalent to the following system

$$\begin{cases} f_0' &= -2qf_0 + \alpha_0, \\ f_1' &= 2qf_1 + \alpha_1, \\ q' &= \frac{1}{2}(f_1 - f_0), \end{cases} \qquad \alpha_0 + \alpha_1 = 1. \tag{3}$$

- The action of $\tilde{W}\left(A_1^{(1)}\right)$ on the symmetric form:

	α_0	α_1	f_0	f_1	q
s_0	$-\alpha_0$	$\alpha_1 + 2\alpha_0$	f_0	$f_1 - 4q\frac{\alpha_0}{f_0} + 2\frac{\alpha_0^2}{f_0^2}$	$q - \frac{\alpha_0}{f_0}$
s_1	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	$f_0 + 4q\frac{\alpha_1}{f_1} + 2\frac{\alpha_1^2}{f_1^2}$	f_1	$q + \frac{\alpha_1}{f_1}$
π	α_1	α_0	f_1	f_0	$-q$

- $C\left(A_1^{(1)}\right) = (c_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$;
- $s_i(\alpha_j) = \alpha_j - c_{ij}\,\alpha_i$.

Symmetric form (2)

$$f_0 = -p + 2q^2 + t, \qquad f_1 = p, \qquad \alpha_0 + \alpha_1 = 1.$$

- Poisson structure:

$\{\,,\,\}$	f_0	f_1	q
f_0	0	$4q$	1
f_1	$-4q$	0	-1
q	-1	1	0

- Denote $\mathrm{ad}_{\{,\}}(\varphi) = \{\varphi, \cdot\}$. Then the action of the $\tilde{W}\left(A_1^{(1)}\right)$ group on the variables for the symmetric form can be written as

$$s_i(\varphi) = \exp\left(-\frac{\alpha_i}{f_i}\,\mathrm{ad}_{\{,\}}(f_i)\right)(\varphi), \qquad \varphi = f_0, f_1, q.$$

For instance,

$$s_0(f_1) = f_1 - 4q\frac{\alpha_0}{f_0} + 2\frac{\alpha_0^2}{f_0^2} = f_1 - \frac{\alpha_0}{f_0}\underbrace{\{f_0, f_1\}}_{4q} + \frac{1}{2}\left(\frac{\alpha_0}{f_0}\right)^2 \underbrace{\{f_0, \{f_0, f_1\}\}}_{4}.$$

Remark. The quantum version of the P_2 equation was written in the paper [Nagoya et al., 2008]:

$$\begin{cases} f_0' &= f_0 q + q f_0 + \alpha_0, \\ f_1' &= -f_1 q - q f_1 + \alpha_1, \\ q' &= f_1 - f_0, \end{cases} \qquad [f_1, f_0] = 2\hbar q, \qquad [f_0, q] = [q, f_1] = \hbar.$$

τ-functions (1)

Hamiltonians

- Note that $H = \frac{1}{2} p \left(p - 2q^2 - t\right) - b\,q = -\frac{1}{2} f_0 f_1 - \alpha_1 q$. Introduce

$$h_0 = -\frac{1}{2} f_0 f_1 - \alpha_1 q, \qquad h_1 = \pi(h_0) = -\frac{1}{2} f_0 f_1 + \alpha_0 q.$$

- Then one can express f_0, f_1, q via h_i:

$$f_1 = -2h_0', \qquad f_0 = -2h_1', \qquad q = h_1 - h_0.$$

- The action of the $\tilde{W}\left(A_1^{(1)}\right)$ group on h_i:

$$s_i(h_j) = h_j,\ i \neq j, \qquad s_j(h_j) = h_j + \frac{\alpha_j}{f_j}, \qquad \pi(h_k) = h_{k+1},\ k \in \mathbb{Z}/2\mathbb{Z}.$$

τ-functions

- Define $h_i = (\ln \tau_i)' = \tau_i' \, \tau_i^{-1}$.
- The Hirota operator is given by $D_t^n \, f(t) \cdot g(t) = \left(\frac{d}{dt} - \frac{d}{ds}\right)^n (f(t)g(s))\Big|_{s=t}$.
- Expressions f_i, q through τ_i read

$$f_i = -\frac{1}{\tau_{i+1}} D_t^2 \tau_{i+1} \cdot \tau_{i+1},\ i \in \mathbb{Z}/2\mathbb{Z}, \qquad q = -\frac{1}{\tau_0 \tau_1} D_t \tau_0 \cdot \tau_1.$$

τ-functions (2)

Bilinear system

- Using the substitution $q = h_1 - h_0 = \left(\ln \tau_1 \tau_0^{-1}\right)'$, $p = -2h_0' = -2(\ln \tau_0)''$, the P_2 system (1) can be written as a bilinear system for the τ-function:

$$\begin{cases} \left(D_t^2 + \frac{1}{2}t\right)\tau_0 \cdot \tau_1 &= 0, \\ \left(D_t^3 + \frac{1}{2}tD_t - \frac{\alpha_0 - \alpha_1}{2}\right)\tau_0 \cdot \tau_1 &= 0. \end{cases}$$

Bäcklund transformations

- Recall that $\quad s_i(h_j) = h_j,\ i \neq j, \qquad s_j(h_j) = h_j + \frac{\alpha_j}{f_j}, \qquad \pi(h_k) = h_{k+1},\ k \in \mathbb{Z}/2\mathbb{Z}.$
- $\tilde{W}\left(A_1^{(1)}\right)$ acts on τ_i by the following rules

$$s_i(\tau_j) = \tau_j,\ i \neq j, \qquad s_k(\tau_k) = -\tau_k^{-1} D_t^2 \tau_{k+1} \cdot \tau_{k+1}, \qquad \pi(\tau_k) = \tau_{k+1},\ k \in \mathbb{Z}/2\mathbb{Z}.$$

- Derive $s_0(\tau_0)$ as an example. On the one hand, $s_0(h_0) = h_0 + \frac{\alpha_0}{f_0}$. On the other hand, $f_0' = -2qf_0 + \alpha_0$ and $q = h_1 - h_0$. Then

$$f_0' f_0^{-1} = -2q + \alpha_0 f_0^{-1} = -2(h_1 - h_0) + s_0(h_0) - h_0 = -2h_1 + h_0 + s_0(h_0),$$

$$(\ln f_0)' = \left(\ln\left(\tau_0\, s_0(\tau_0)\, \tau_1^{-2}\right)\right)' \qquad \Leftrightarrow \qquad f_0 = \underbrace{c_0}_{=1}\, \tau_0\, s_0(\tau_0)\, \tau_1^{-2}.$$

Since $f_0 = -\tau_1^{-1} D_t^2 \tau_1 \cdot \tau_1$, we have $s_0(\tau_0) = -\tau_0^{-1} D_t^2 \tau_1 \cdot \tau_1$.

τ-functions on the lattice (1)

- The translation $T_1 = (\pi s_1)^1 = rs$:

$$T_1(\alpha_0) = \alpha_0 + 1, \quad T_1(\alpha_1) = \alpha_1 - 1, \qquad T_1(\tau_0) = \tau_1, \quad T_1(\tau_1) = f_0 \frac{\tau_1^2}{\tau_0},$$

$$T_1(f_0) = f_1 - 4q\frac{\alpha_0}{f_0} + 2\left(\frac{\alpha_0}{f_0}\right)^2, \quad T_1(f_1) = f_0, \quad T_1(q) = -q + \frac{\alpha_0}{f_0}.$$

- Note $\quad T_n(\alpha_0) = \alpha_0 + n, \qquad T_n(\alpha_1) = \alpha_1 - n, \qquad n \in \mathbb{N}.$
- Similarly to $T_1(\tau_0) = \tau_1$, introduce $T_n(\tau_0) = \tau_n$, where $n \in \mathbb{N}$. Then $\tilde{W}\left(A_1^{(1)}\right)$ acts on τ_n by reflections w.r.t. $n = 1$, $n = 0$, and $n = \frac{1}{2}$ respectively:

$$s_0(\tau_n) = \tau_{2-n}, \qquad s_1(\tau_n) = \tau_{-n}, \qquad \pi(\tau_n) = \tau_{1-n}.$$

- Applying T_n to the bilinear system, we will get

$$\begin{cases} \left(D_t^2 + \frac{1}{2}t\right)\tau_n \cdot \tau_{n+1} &= 0, \\ \left(D_t^3 + \frac{1}{2}tD_t - \frac{\alpha_0 - \alpha_1}{2} - n\right)\tau_n \cdot \tau_{n+1} &= 0. \end{cases}$$

- Since $s_1(\tau_1) = \tau_{-1}$ and $s_1(\tau_1) = -\tau_1^{-1} D_t^2 \tau_0 \cdot \tau_0$, the action of T_n on the equation $\tau_{-1}\tau_1 = -D_t^2\tau_0 \cdot \tau_0$ leads to the Toda lattice equation:

$$\tau_{n-1}\tau_{n+1} = -2\left(\tau_n''\tau_n - {\tau_n'}^2\right), \qquad n \in \mathbb{Z}.$$

τ-functions on the lattice (2)

$$\tau_{n-1}\tau_{n+1} = -2\left(\tau_n''\tau_n - {\tau_n'}^2\right), \qquad n \in \mathbb{Z}. \tag{4}$$

- Let us call τ_0 and τ_1 *initial conditions* and try to express τ_n via them.
- For instance,

$$\tau_2 = T_2(\tau_0) = \pi s_1 \pi s_1(\tau_0) = \pi s_1 \pi(\tau_0) = \pi s_1(\tau_1) = \pi\left(f_1\frac{\tau_0^2}{\tau_1}\right) = f_0\frac{\tau_1^2}{\tau_0},$$

$$\tau_3 = T_3(\tau_0) = \pi s_1(\tau_2) = s_0\pi\left(f_0\frac{\tau_1^2}{\tau_0}\right) = s_0\left(f_1\frac{\tau_0^2}{\tau_1}\right) = \left(f_1 - 4q\frac{\alpha_0}{f_0} + 2\left(\frac{\alpha_0}{f_0}\right)^2\right)\frac{\tau_1^3}{\tau_0^2},$$

$$\vdots$$

- By induction, one can show there exist **rational** functions $\varphi_n(f_0, f_1, q; \alpha_0, \alpha_1)$ such that

$$\begin{gathered}\tau_n = T_n(\tau_0) = \varphi_n \tau_1^n \tau_0^{1-n}, \\ \varphi_{n+1} = T_1(\varphi_n) f_0^n, \quad \varphi_{n-1} = T_{-1}(\varphi_n) f_1^{1-n},\end{gathered} \qquad n \in \mathbb{Z}.$$

- The Toda equation (4) together with the definition of τ_n via φ_n implies the following recurrence equation for φ_n

$$\varphi_{n-1}\varphi_{n+1} = -2\left(\varphi_n''\varphi_n - {\varphi_n'}^2\right) + \left(nf_0 - (n-1)f_1\right)\varphi_n^2, \qquad \varphi_0 = \varphi_1 = 1.$$

- The T_n-action on $q = h_1 - h_0 = \left(\ln \tau_1 \tau_0^{-1}\right)'$ gives $q_n = T_n(q) = \left(\ln \tau_{n+1}\tau_n^{-1}\right)'$.

Rational solutions (1)

$$\varphi_{n-1}\varphi_{n+1} = -2\left(\varphi_n''\varphi_n - {\varphi_n'}^2\right) + \left(nf_0 - (n-1)f_1\right)\varphi_n^2, \qquad \varphi_0 = \varphi_1 = 1. \qquad (5)$$

- The symmetric form (3) of the P_2 equation,

$$\begin{cases} f_0' &= -2qf_0 + \alpha_0, \\ f_1' &= 2qf_1 + \alpha_1, \\ q' &= \frac{1}{2}(f_1 - f_0), \end{cases} \qquad \alpha_0 + \alpha_1 = 1,$$

has a **unique** trivial solution (also called *a seed solution*)

$$(f_0, f_1, q; \alpha_0, \alpha_1) = \left(\frac{t}{2}, \frac{t}{2}, 0; \frac{1}{2}, \frac{1}{2}\right).$$

- Then $nf_0 - (n-1)f_1 = \frac{1}{2}t$.
- The recurrence relation (5) becomes

$$\varphi_{n-1}\varphi_{n+1} = -2\left(\varphi_n''\varphi_n - {\varphi_n'}^2\right) + \frac{1}{2}t\varphi_n^2, \qquad \varphi_0 = \varphi_1 = 1,$$

where φ_n are *the Yablonskii-Vorobiev polynomials.*

- As $q_n = \left(\ln \tau_{n+1}\,\tau_n^{-1}\right)' = q + \left(\ln \varphi_{n+1}\,\varphi_n^{-1}\right)'$ and $q = y$, rational solutions of P_2 are

$$y_n = \left(\ln \varphi_{n+1}\,\varphi_n^{-1}\right)'.$$

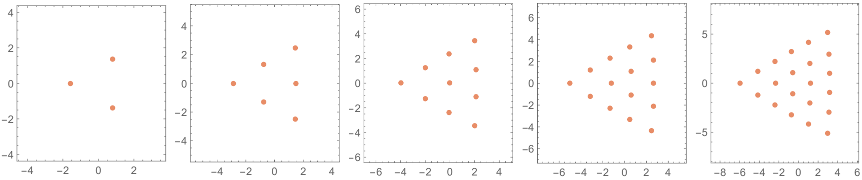
Rational solutions (2)

$$\varphi_{n-1}\varphi_{n+1} = -2\left(\varphi_n''\varphi_n - {\varphi_n'}^2\right) + \tfrac{1}{2}t\varphi_n^2, \qquad \varphi_0 = \varphi_1 = 1. \tag{6}$$

- $\varphi_{-2} = \frac{1}{8}\left(t^3+4\right) = \varphi_3, \qquad \varphi_{-1} = \frac{1}{2}t = \varphi_2, \qquad \ldots$

$$y_n = \left(\ln \varphi_{n+1}\,\varphi_n^{-1}\right)', \qquad y_{-n} = -y_n. \tag{7}$$

- $y_0 = \left(\ln \varphi_1\varphi_0^{-1}\right)' = 0, \qquad b = \frac{1}{2},$
- $y_1 = \left(\ln \varphi_2\varphi_1^{-1}\right)' = \dfrac{1}{t}, \qquad b = \frac{3}{2},$
- $y_2 = \left(\ln \varphi_3\varphi_2^{-1}\right)' = \dfrac{2\left(t^3-2\right)}{t\left(t^3+4\right)}, \qquad b = \frac{5}{2},$
- $\ldots$



$n=2$ $\qquad$ $n=3$ $\qquad$ $n=4$ $\qquad$ $n=5$ $\qquad$ $n=6$

Figure: Plots of Yablonskii-Vorobiev polynomials roots $\tilde{\varphi}_n(z) = 0$, $n = 2, 3, 4, 5, 6$.

P_2 solutions: a determinant structure (1)

- The Toda equation

$$u_n'' = e^{u_{n-1}-u_n} - e^{u_n - u_{n+1}}, \qquad u_n = u_{-n-1}$$

after the substitution $u_n = \ln \tau_{n-1}\tau_n^{-1}$ becomes

$$\tau_{n-1}\tau_{n+1} = \tau_n''\tau_n - \tau_n'^2, \qquad n \in \mathbb{Z}.$$

- The substitution $\sigma_n = \tau_n\tau_0^{-1}$ transforms the latter equation to the form

$$\sigma_{n-1}\sigma_{n+1} = \sigma_n''\sigma_n - \sigma_n'^2 + \varphi\psi\sigma_n^2, \qquad \sigma_{-1} = \psi, \quad \sigma_0 = 1, \quad \sigma_1 = \phi. \tag{8}$$

Theorem. [Kajiwara et al., 1999] Let $a_n = a_n(t)$ and $b_n = b_n(t)$, $n \in \mathbb{N}$, are defined by

$$a_n = a_{n-1}' + \psi \sum_{i+j=n-2} a_i a_j, \quad a_0 = \varphi; \qquad b_n = b_{n-1}' + \varphi \sum_{i+j=n-2} b_i b_j, \quad b_0 = \psi.$$

Let σ_n are $n \times n$ determinants in the Hankel form

$$\sigma_n = \begin{cases} det\left(a_{i+j-2}\right)_{i,j\le n}, & n > 0; \\ 1, & n = 0; \\ det\left(b_{i+j-2}\right)_{i,j\le |n|}, & n < 0. \end{cases} \tag{9}$$

Then σ_n satisfy (8).

P_2 solutions: a determinant structure (2)

$$\sigma_{n-1}\sigma_{n+1} = \sigma_n''\sigma_n - {\sigma_n'}^2 + \varphi\psi\sigma_n^2, \qquad \sigma_{-1} = \psi, \qquad \sigma_0 = 1, \qquad \sigma_1 = \phi. \tag{10}$$

Theorem. [Kajiwara et al., 1999] Let $a_n = a_n(t)$ and $b_n = b_n(t)$, $n \in \mathbb{N}$, are defined by

$$a_n = a_{n-1}' + \psi \sum_{i+j=n-2} a_i a_j, \qquad a_0 = \varphi; \qquad\qquad b_n = b_{n-1}' + \varphi \sum_{i+j=n-2} b_i b_j, \qquad b_0 = \psi.$$

Let σ_n are $n \times n$ determinants in the Hankel form (9). Then σ_n satisfy (10).

Sketch of the proof.

- Set $D = \det X_{n+1,n+1}$. Let $D\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix}$ be a determinant of the matrix obtained from the $X_{n+1,n+1}$ matrix by removing i_l rows and j_l columns, $l = 1, \dots, k$.
- *The Lewis Carroll formula*

$$D\, D\begin{pmatrix} n & n+1 \\ n & n+1 \end{pmatrix} = D\begin{pmatrix} n \\ n \end{pmatrix} D\begin{pmatrix} n+1 \\ n+1 \end{pmatrix} - D\begin{pmatrix} n \\ n+1 \end{pmatrix} D\begin{pmatrix} n+1 \\ n \end{pmatrix}.$$

- $\sigma_{n+1} = D, \qquad \sigma_n = D\begin{pmatrix} n+1 \\ n+1 \end{pmatrix}, \qquad \sigma_{n-1} = D\begin{pmatrix} n & n+1 \\ n & n+1 \end{pmatrix}.$
- $\sigma_n' = D\begin{pmatrix} n \\ n+1 \end{pmatrix} = D\begin{pmatrix} n+1 \\ n \end{pmatrix}.$
- $\sigma_n'' + \varphi\,\psi\sigma_n = D\begin{pmatrix} n \\ n \end{pmatrix}.$

P_2 solutions: a determinant structure (3)

- The transformations

$$t = (-4)^{\frac{1}{3}} z, \qquad y(t) = (-4)^{-\frac{1}{3}} u(z), \qquad b = -a,$$

bring the P_2 equation to the form

$$u'' = 2u^3 - 4zu + 4(a + \tfrac{1}{2}). \qquad P_2[a]$$

Theorem. [Joshi et al., 2004] Let $\psi = \psi(z)$ and $\varphi = \varphi(z)$ satisfy the equations

$$\psi''\psi^{-1} = \varphi''\varphi^{-1} = -2\psi\varphi + 2z, \qquad \varphi'\psi - \varphi\psi' = -2a.$$

Then

- $u_0 = (\ln\varphi)'$ is a solution of $P_2[a]$;
- $u_{-1} = (\ln\psi)'$ is a solution of $P_2[a-1]$;
- $u_n = \left(\ln \sigma_{n+1}\sigma_n^{-1}\right)'$ is a solution of $P_2[a+n]$.

Sketch of the proof.

- In the case when $n = 0$ and $n = -1$, the proof is a straightforward computation.
- When n is arbitrary, we have reformulated Theorem in [Kajiwara et al., 1999].

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