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CLASSIFICATION OF INTEGRABLE EVOLUTION EQUATIONS

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Abstract

This survey considers the problems of classification of both differential and difference evolution equations. We point out a class of equations containing an arbitrary function of two arguments, that are analogous to the Burgers equation. A complete list is given of equations of the form $u_t = u_{xxx} + f(u, u_x, u_{xx})$, analogous to the Korteweg–deVries equation. New examples are found of equations analogous to the difference variant of the Korteweg–deVries equation. The classification is based on the conditions for nontriviality of the Lie–Bäcklund algebra. We also consider modifications of these conditions, related to the conservation laws and Bäcklund transformations.

Contents

Introduction

In our survey we present results of papers [1–8]. In these papers new examples were found and principles for the classification of integrable nonlinear equations were pointed out.

In §§1, 2 we consider scalar evolution equations

$$u_t = F(u, u_1, \ldots, u_m), \qquad m \geqslant 2 \tag{0.1}$$

where $u_i \overset{\text{def}}{=} \partial^i u/\partial x^i$. The classification of Eqs. (0.1) is based on an investigation of the solvability of the operator relation

$$L_t - [F_*, L] = 0 \tag{0.2}$$

Here $F_* = \sum_{i=0}^{m}(\partial F/\partial u_i)D^i$, $D \overset{\text{def}}{=} \partial/\partial x$, $L_t \overset{\text{def}}{=} [\partial/\partial t, L]$. We require that there exists a series

$$L = \sum_{i=-\infty}^{r} f_i D^i, \qquad f_i = f_i(u, u_1, \ldots, u_{n_i}) \tag{0.3}$$

satisfying condition (0.2) on solutions of Eq. (0.1). For example, for the Burgers equation

$$u_t = u_2 + uu_1 \tag{0.4}$$

the relation (0.2) is satisfied by $L = D + \frac{1}{2}u + \frac{1}{2}u_1 D^{-1}$. In fact, in this case the left side of (0.2) has the form

$$\frac{1}{2}(u_t - u_2 - uu_1) + \frac{1}{2}\frac{\partial}{\partial x}(u_t - u_2 - uu_1)D^{-1}$$

and therefore vanishes on solutions of (0.4).

Along with (0.2) we consider the equation

$$S_t + F_*^t S + SF_* = 0 \tag{0.5}$$

where F_*^t is the differential operator formally adjoint to F_*. This equation is closely related to Eq. (0.2). Namely, if S_1, S_2 are solutions of (0.5), then $L = S_1^{-1}S_2$ is a solution of (0.2). For example, for the Korteweg–deVries equation

$$u_t = u_3 + uu_1 \tag{0.6}$$

the solutions of (0.5) are (cf. [9])

$$S_1 = \left(D^3 + \tfrac{2}{3}uD + \tfrac{1}{3}u_1\right)^{-1}, \qquad S_2 = D^{-1}$$

and so $L = D^2 + \frac{2}{3}u + \frac{1}{3}u_1 D^{-1}$ satisfies relation (0.2). The choice of equation (0.2) as fundamental is explained by the fact that, unlike the set of solutions of Eq. (0.5), the set $\hat{A}(F)$ of solutions of (0.2) is closed under multiplication and extraction of roots of degree r from solutions of the form (0.3).

Let us enumerate the basic results of §1. Theorem 1.7 asserts that if the Eq. (0.1) has an infinite Lie–Bäcklund algebra, then Eq. (0.2) is solvable. Theorem 1.12 guarantees the solvability of (0.2) and (0.5) for the case when Eq. (0.1) has an infinite series of local conservation laws. Theorem 1.9 shows that if Eq. (0.2) is solvable, the first order operator L in $\hat{A}(F)$ generates a series of conservation laws for Eq. (0.1):

$$\frac{\partial}{\partial t}(\operatorname{res} L^{-1}) \in \operatorname{Im} D, \qquad \operatorname{res}(L^{-1}L_t) \in \operatorname{Im} D, \qquad\qquad (0.7)$$
$$\frac{\partial}{\partial t}(\operatorname{res} L^{k}) \in \operatorname{Im} D, \qquad k \in \mathbb{N}$$

where $\operatorname{res}(\sum a_i D^i) \overset{\text{def}}{=} a_{-1}$. It turns out that several of the first few conservation laws (0.7) can be written in terms of F. For example, the first conservation law has the form $\partial/\partial t(\partial F/\partial u_m)^{-1/m} \in \operatorname{Im} D$, the second is expressed in the form $\partial/\partial t((\partial F/\partial u_{m-1})/(\partial F/\partial u_m)) \in \operatorname{Im} D$, etc. The existence for Eq. (0.1) of the conservation laws (0.7) imposes strict conditions on the function F.

Some of the conservation laws (0.7) may turn out to be trivial. For example, in the case of the Burgers equation

$$L = D + \tfrac{1}{2}u + \tfrac{1}{2}u_1 D^{-1} = D(D + \tfrac{1}{2}u)D^{-1}$$

and thus

$$\operatorname{res} L^k = \operatorname{res} D(D + \tfrac{1}{2}u)^k D^{-1} = \tfrac{1}{2}D\left[(D + \tfrac{1}{2}u)^{k-1}(u)\right] \in \operatorname{Im} D$$

In Sec. 1.4 we show that the conditions

$$\left(\frac{\partial F}{\partial u_{m-1}} \Big/ \frac{\partial F}{\partial u_m}\right) \in \operatorname{Im} D + \mathbb{C}, \qquad \operatorname{res} L^k \in \operatorname{Im} D + \mathbb{C}, \qquad k \in \mathbb{N} \qquad (0.8)$$

which indicate triviality of the corresponding conservation laws of the series (0.7), are satisfied for Eq. (0.1), being reduced by the differential substitution $v = (u, \ldots, u_k)$ to linear equations with constant coefficients.

When only some of the first of the conditions (0.8) are violated, one can introduce auxiliary variables $\overset{(i)}{u}$, $i = 0, 1, 2, \ldots,$ satisfying the relations

$$D\overset{(0)}{u} = \frac{\partial F}{\partial u_{m-1}} \Big/ \frac{\partial F}{\partial u_m}, \qquad D\overset{(i)}{u} = \operatorname{res} L^i, \quad i = 1, 2, \ldots$$

and seek a substitution reducing Eq. (0.1) to a linear equation of the form

$$v = \varphi\left(\overset{(k)}{u}, \ldots, \overset{(1)}{u}, \overset{(0)}{u}, u_1, \ldots, u_n\right)$$

For example, for Eq. (0.4), the first of the conditions (0.8) is violated, since $(\partial F/\partial u_1)/(\partial F/\partial u_2) = u$. Introducing the auxiliary variable $\overset{(0)}{u} = D^{-1}u$, it is not difficult to find the substitution $v = \varphi(\overset{(0)}{u})$ reducing the Burgers equation to the heat conduction equation.

If the series (0.7) contains an infinite number of nontrivial conservation laws, then there is apparently no chance of finding a substitution connecting Eq. (0.1) with a linear equation. In this case one can try to apply the inverse scattering technique to Eq. (0.1), or look for a substitution that reduces it to an equation that is integrable by the inverse scattering method.

In §2 we consider the problem of enumerating those equation (0.1) of order $m = 2, 3$ that satisfy conditions (0.7). More precisely, by expressing the first coefficients of the operator $L = \alpha D + \beta + \gamma D^{-1} + \cdots$ in terms of the function F, we study an overdetermined system of partial differential equations, equivalent to several of the first conditions in (0.7).

For $m = 2$ we give a complete list of equations satisfying the first three conditions of (0.7). For all the equations in this list the fourth condition is satisfied identically. The list contains equations whose right sides depend on functional parameters. For certain of these equations the question of the solvability of Eq. (0.2) remains open at present. We give two examples for which one has succeeded in finding a solution of (0.2). The first equation

$$u_t = u_1^{-2}u_2 + a(u) + b(u)u_1$$

contains arbitrary functions a and b of one variable. The solution of (0.2) is the differential operator

$$L = u_1^{-2}D^2 - u_1^{-3}(3u_2 + au_1^2)D + u_1^{-4}(3u_2^2 + au_2u_1^2 - u_3u_1).$$

The second equation

$$u_t = -\frac{1}{f_1^2u_2 + f_0f_1u_1} + c(u, u_1),$$

where $f_0 \equiv \partial f/\partial u$, $f_1 \equiv \partial f/\partial u_1$, contains an arbitrary function f of two variables. The function c is found from the relation

$$\left(f_0u_1\frac{\partial}{\partial u_1} - f_1u_1\frac{\partial}{\partial u} + f_0\right)c + \frac{\partial f_1}{\partial u_1}f_1^{-2} + 2(f_1u_1)^{-1} = 0.$$

In this case

$$L = (D(f))^{-1}D + f_0(f_1D(f))^{-1} - (f_1u_1)^{-1}.$$

In addition to the list of second order equations, §2 contains a complete list of equations of the form

$$u_t = u_3 + \varphi(u, u_1, u_2)$$

satisfying the first four conditions of (0.7). The list contains, for example, the equation

$$u_t = u_3 - \tfrac{3}{2}u_1^{-1}u_2^2, \tag{0.9}$$

which is invariant under the fractional linear transformations $u \leftrightarrow (\alpha u + \beta)(\gamma u + \delta)^{-1}$. The substitution

$$v = \tfrac{1}{2}\left(u_1^{-1}u_3 - \tfrac{3}{2}u_1^{-2}u_2^2\right) \tag{0.10}$$

transfers the solution of (0.9) to a solution of the Korteweg–deVries equation, $v_t = v_3 - 6vv_1$. Since the right side of (0.10) is the Schwartzian derivative of the function u, the problem of inversion of the transformation (0.10) is well understood. The inverse transformation to (0.10) has the form $u = \varphi_1/\varphi_2$, where φ_1, φ_2 are a basis of solutions of the equation $(D^2 - v)\varphi = 0$, normalized by the condition $\varphi_1'\varphi_2 - \varphi_1\varphi_2' = 1$. The transition to another normalized basis results in a fractional-linear transformation of u.

Without comment we give another of the 17 equations in the list:

$$u_t = u_3 - \frac{3}{4}\frac{u_2^2}{u_1 + 1} + 3(u_1 + 1)^{1/2}u_2\frac{1}{\cos u} + 3u_2(1 + u_1)\operatorname{tg} u$$
$$+ 6u_1(1 + u_1)^{3/2}\frac{\operatorname{tg} u}{\cos u} + 3u_1(1 + u_1)(2 + u_1)\operatorname{tg}^2 u + 2(1 + u_1)^3 \tag{0.11}$$

In §3 we consider discrete evolution equations

$$\frac{d}{dt}u_k = D^k F(u_n, u_{n+1}, \dots, u_m), \qquad k \in \mathbb{Z} \tag{0.12}$$

where

$$D^k(F(u_n, \dots, u_m)) \stackrel{\text{def}}{=} F(u_{n+k,m}, u_{m+k})$$

We note that from the point of view of differential algebra Eq. (0.1) is an abbreviated expression for the infinite system of equations

$$\frac{\partial u_k}{\partial t} = D^k(F(u, \dots, u_m))$$

analogous to (0.12). The only essential difference of differential equations from the discrete case is that in the first case D is a derivation, and in the second it is an automorphism. The distinction in the properties of the operator D makes impossible an automatic transfer of the results of §1 to the case of discrete equations. The specific features of the discrete case are discussed in §3.1. In §3 we preserve the notation of §1, hoping that this does not cause any misunderstandings. In particular, the system (0.12) is written in abbreviated form as

$$u_t = F(u_n, \ldots, u_m)$$

In §3.2 for the example of equations of the form

$$u_t = F(u_{-1}, u, u_1)$$

it is shown that the first conditions for solvability of relations analogous to (0.2), (0.5) allow us to distinguish a quite narrow class of equations that deserve additional study. Characteristic examples of equations of this class are

$$u_t = (\alpha u^3 + \beta u^2 + \gamma u + \delta)\left(\frac{1}{u - u_1} + \frac{1}{u_{-1} - u}\right) \qquad (0.13)$$

$$u_t = \frac{(u_1 - u)(u - u_{-1})}{u_1 - u_{-1}} \qquad (0.14)$$

$$u_t = \frac{u_1 u_{-1} + u^2 - 1}{u_1 - u_{-1}} \qquad (0.15)$$

An interesting problem is the comparison of the lists of equations from §§2, 3. For example, Eq. (0.14) is similar in its properties to Eq. (0.9). Namely, Eq. (0.14) is invariant under fractional-linear transformations and reduces to the difference analog of the Korteweg–deVries equation $v_t = v(v_1 - v_{-1})$ under the substitution

$$v = \frac{(u_3 - u_2)(u_1 - u)}{(u_3 - u_1)(u_2 - u)}$$

It is necessary to remark that in §§1–3 all functions are assumed to be analytic. The presentation is given in a language close to that of differential algebra. This language proves convenient, since the proposed classification is actually algebraic.

In conclusion we wish to express our gratitude to S. I. Svinolupov and R. I. Yamilov for numerous useful discussions of the contents of this survey.

§1 Necessary Conditions for Existence of a Lie–Bäcklund Algebra and Conservation Laws

1.1 Lie–Bäcklund algebra

Let $u, u_1, u_2, \ldots$ be an infinite set of independent variables. Denote by $\mathscr{F}$ the algebra of all functions depending on a finite number of variables from this set. If $\partial/\partial u_n\, f(u, u_1, \ldots, u_n) \neq 0$, then the number n will be called the order of f and denoted by ord f. It is clear that $D \overset{\text{def}}{=} \sum_{i=0}^{\infty} u_{i+1}(\partial/\partial u_i)$ is a derivation of $\mathscr{F}$, mapping u_k into u_{k+1}, where $k = 0, 1, 2, \ldots, u_0 = u$.

We associate with each function $f \in \mathscr{F}$ the derivation $\partial_f : \mathscr{F} \to \mathscr{F}$, taking u into f and commuting with D. On any function $g(u, \ldots, u_k)$ the derivation ∂_f acts according to the formula

$$\partial_f(g) = g_*(f) \tag{1.1}$$

where $g_* \overset{\text{def}}{=} \sum_{i=0}^{k}(\partial g/\partial u_i)D^i$. Derivations of the form (1.1) are said to be evolutionary. Evolutionary derivations form a Lie algebra. It is not difficult to verify that $[\partial_f, \partial_g] = \partial_h$, where

$$h = g_*(f) - f_*(g) \overset{\text{def}}{=} \{f, g\} \tag{1.2}$$

Lemma 1.1 If $h = \{f, g\}$, then

$$\left[\partial_f - f_*, \partial_g - g_*\right] = \partial_h - h_*$$

Proof From the definition of the operation * it is clear that

$$(\alpha\beta)_* = \alpha\beta_* + \beta\alpha_*, \qquad (D\alpha)_* = D \circ \alpha_*, \qquad \forall \alpha, \beta \in \mathscr{F} \tag{1.3}$$

Using these relations, we find that

$$h = g_*(f) - f_*(g) \Rightarrow h_* = [g_*, f_*] + [f_*, \partial_g] + [\partial_f, g_*].$$

The Lie–Bäcklund algebra $A(F)$ of Eq. (0.1) is the set of all functions $f \in \mathscr{F}$ such that $[\partial_F, \partial_f] = 0$. The last equality is equivalent to the equation

$$(\partial_F - F_*)f = 0 \tag{1.4}$$

It is clear that $A(F)$ is a Lie algebra with respect to the bracket (1.2). The functions u_1 and F always belong to $A(F)$.

The presence in the algebra $A(F)$ of elements arbitrarily high order is a characteristic feature of equations analogous to the Korteweg–deVries equation (0.6). It is known (cf. [1], [10]) that all elements of the Lie–Bäcklund algebra of this equation can be found from the recursion relation

$$\overset{(n+1)}{f} = L\overset{(n)}{f}, \qquad \overset{(1)}{f} = u_1$$

where L is the operator given in the Introduction.

From Lemma 1.1 it follows that for any function $f \in A(f)$ the equality

$$\partial_F(f_*) - [F_*, f_*] = \partial_f(F_*) \tag{1.5}$$

is satisfied. Here and in the sequel, for any function $h \in \mathscr{F}$,

$$\partial_h\left(\sum a_i D^i\right) \overset{\text{def}}{=} \left[\partial_h, \sum a_i D^i\right] = \sum \partial_h(a_i) D^i$$

Suppose that $\operatorname{ord} f = n \geqslant 2$. Comparing coefficients of D^j, $j = m + n - 1, m + n - 2, \ldots, m + 1$ in (1.5) and using the fact that the order of the differential operator $\partial_f(F_*)$ is no greater than m, we get

$$m \frac{\partial F}{\partial u_m} D\left(\frac{\partial f}{\partial u_k}\right) - k \frac{\partial f}{\partial u_k} D\left(\frac{\partial F}{\partial u_m}\right) = \overset{(n)}{\varphi}_k\left(\frac{\partial f}{\partial u_{k+1}}, \ldots, \frac{\partial f}{\partial u_n}, F\right),$$

$$k = n, n-1, \ldots, 2 \tag{1.6}$$

where $\overset{(n)}{\varphi}_k$ are differential polynomials. From relations (1.6), we can express $\partial f/\partial u_n, \partial f/\partial u_{n-1}, \ldots, \partial f/\partial u_2$ in terms of F. In particular, for $m \geqslant 2$

$$\frac{\partial f}{\partial u_n} = \text{const}\left(\frac{\partial F}{\partial u_m}\right)^{n/m},$$

since the right side is equal to zero in the first of Eqs. (1.6).

THEOREM 1.2 Let $F(u, \ldots, u_m)$ be a polynomial, $m \geqslant 2$. Then, if $\partial F/\partial u_m = \text{const}$, any element $f(u, \ldots, u_n) \in A(F)$ has the form $f(u, \ldots, u_n) = f_1(u, \ldots, u_n) + f_2(u)$, where f_1 is a polynomial. If $\partial F/\partial u_m = \text{const}$, $\partial F/\partial u_{m-1} = \text{const}$, then f is a polynomial.

Proof Without loss of generality we may assume that $\partial F/\partial u_m = 1$. In that case the Eqs. (1.6) have the form

$$D\left(\frac{\partial f}{\partial u_k}\right) = \overset{(n)}{\varphi}_k\left(\frac{\partial f}{\partial u_{k+1}}, \ldots, \frac{\partial f}{\partial u_n}, F\right), \qquad k = n, n-1, \ldots, 1 \tag{1.7}$$

We note that the number of equations has increased by one, since the order of the operator $\partial_f(F_*)$ on the right side of (1.5) has been reduced by unity. Using the fact that the polynomial nature of $D(\partial f/\partial u_k)$ implies the same for $\partial f/\partial u_k$, we find from (1.7) by induction that $\partial f/\partial u_n, \ldots, \partial f/\partial u_1$ are polynomials. The first statement of the theorem is thus proved. It remains to note that for $\partial F/\partial u_{m-1} = \text{const}$ there is an additional decreasing of the order of the operator on the right side of (1.7), and the relations (1.7) are satisfied for $k = n, n-1, \ldots, 0$.

The example of the equation $u_t = u_2 + u_1^2$, whose Lie–Bäcklund algebra contains the element e^u, shows the importance of the requirement $\partial F/\partial u_{m-1} = \text{const}$.

We note that the structure of the algebra $A(F)$ depends essentially on the subalgebra A_1 consisting of elements of $A(F)$ of order no higher than unity.

Hypothesis The subalgebra A_1 is an ideal. The algebra $A(F)$ decomposes into the semidirect product $A(F) = A_1 \times A_2$, where the subalgebra A_2 is commutative.

1.2 The algebra $\hat{A}(F)$

Dropping $\partial_f(F_*)$ on the right side in (1.5), which does not affect the form of Eqs. (1.6), we come to the equation

$$\left[\partial_F - F_*, L\right] = 0 \tag{1.8}$$

We shall be interested in the solvability of Eq. (1.6) in the algebra $\mathcal{F}((D^{-1}))$ of formal series of the form $L = \sum_{i=-\infty}^{n} a_i D^i$ with coefficients in $\mathcal{F}$. We recall that the operation of multiplication in $\mathcal{F}((D^{-1}))$ is defined by the formula

$$D^n \circ a = \sum_{k=0}^{\infty} \frac{1}{k!} n(n-1)\cdots(n-k+1) D^k(a) D^{n-k}, \qquad n \in \mathbb{Z}$$

For brevity we shall call the elements of $\mathcal{F}((D^{-1}))$ operators.

It is obvious that the set of solutions $\hat{A}(F)$ of Eq. (1.8) form a subalgebra in $\mathcal{F}((D^{-1}))$. For any operator $L \in \mathcal{F}((D^{-1}))$ of order n, there exists (cf., for example, [11]) an operator $L^{1/n} \in \mathcal{F}((D^{-1}))$ such that $(L^{1/n})^n = L$. The operator $L^{1/n}$ is unique up to multiplication by an nth root of unity. Its coefficients are differential polynomials in the coefficients of the original operator L.

LEMMA 1.3 Let $L \in \hat{A}(F)$, $\operatorname{ord} L = n$. Then $L^{1/n} \in \hat{A}(F)$.

Proof Set $M = L^{1/n}$. Substituting M^n in Eq. (1.8) and denoting $[\partial_F - F_*, M]$ by R, we get

$$RM^{n-1} + MRM^{n-2} + \cdots + M^{n-1}R = 0$$

Equating the leading coefficient in the operator on the left side of this equation to zero, we find that the leading coefficient R is equal to zero, i.e., $R = 0$.

LEMMA 1.4 Let $L \in \hat{A}(F)$, $\operatorname{ord} L = n \neq 0$. Then $\hat{A}(F) = \mathbb{C}((L^{-1/n}))$.

Proof Let $\tilde{L} \in \hat{A}(F)$, $\operatorname{ord} \hat{L} = k$. Equating in the relation $\partial_F(\tilde{L}) = [F_*, \tilde{L}]$ the coefficients of D^{k+m-1}, we find that the coefficient of the operator $\tilde{L}$ is equal to $c_1(\partial F/\partial u_m)^{k/m}$, $c_1 \in \mathbb{C}$. From Lemma 1.3 it follows that $L^{k/n} \in \hat{A}(F)$, and therefore the leading coefficient in

$L^{k/n}$ is equal to $c_2(\partial F/\partial u_m)^{k/m}$. Consequently the order of the operator $\tilde{L}$ can be lowered by subtracting the operator $c_1 c_2^{-1} L^{k/n}$. Continuing this process, we arrive at the expansion

$$\tilde{L} = \sum_{i=-\infty}^{k} \tilde{c}_i L^{i/n}, \qquad \tilde{c}_i \in \mathbb{C}$$

We denote by $\hat{A}_j \equiv \hat{A}_j(F)$ the set of all operators $L \in \mathscr{F}((D^{-1}))$ such that

$$\operatorname{ord}\left[\partial_F - F_*, L\right] \leqslant m + \operatorname{ord} L - j - 1$$

It is clear that $\mathscr{F}((D^{-1})) = \hat{A}_0 \supset \hat{A}_1 \supset \hat{A}_2 \supset \cdots \supset \hat{A}_\infty = \hat{A}(F)$.

Lemma 1.5 An operator L of order n belongs to $\hat{A}_j(F)$ if and only if $L^{1/n}$ belongs to $\hat{A}_j(F)$.

Proof We must verify that

$$\operatorname{ord}\left[\partial_F - F_*, L\right] \leqslant m + n - j - 1 \Leftrightarrow \operatorname{ord}\left[\partial_F - F_*, L^{1/n}\right] \leqslant m - j$$

This follows from the formula

$$\begin{aligned}\left[\partial_F - F_*, L\right] = {} & \left[\partial_F - F_*, L^{1/n}\right] L^{(n-1)/n} \\ & + L^{1/n}\left[\partial_F - F_*, L^{1/n}\right] L^{(n-2)/n} \\ & + \cdots + L^{(n-1)/n}\left[\partial_F - F_*, L^{1/n}\right].\end{aligned}$$

We shall say that Eq. (1.8) is solvable in $\hat{A}_j$ if $\hat{A}_j$ contains at least one operator L of nonzero order. In this case, by virtue of Lemma 1.5, $\hat{A}_j$ contains elements of any order. Moreover, following the proof of Lemma 1.4, one can show that an arbitrary operator $\tilde{L} \in \hat{A}_j$ admits the representation

$$\tilde{L} = \sum_{k+1-j}^{k} c_i L^{i/n} + \sum_{-\infty}^{k-j} b_i D^i, \qquad c_i \in \mathbb{C}, \quad b_i \in \mathscr{F}, \tag{1.9}$$

where $n = \operatorname{ord} L$. The expansion (1.9) is unique.

LEMMA 1.6 Suppose that the Lie–Bäcklund algebra $A(F)$ contains an element f of order n. Then $f_* \in \hat{A}_{n-1}$.

The assertion of Lemma 1.6 follows from the formula (1.5).

THEOREM 1.7 If $A(F)$ contains elements of arbitrarily high order, then there exists an operator $L \in \mathscr{F}((D^{-1}))$ of any fixed order r, satisfying relation (1.8).

Proof From Lemmas 1.5, 1.6, it follows that for any $i > 0$ the set $\hat{A}_i$ contains an operator of order r,

$$L_i = \overset{(i)}{a}_1 D^r + \overset{(i)}{a}_2 D^{r-1} + \cdots .$$

The leading coefficient of the operator L_i is proportional to $(\partial F/\partial u_m)^{r/m}$. We shall assume, replacing if necessary L_i by $c_i L_i$, that all the operators L_i, $i \geqslant 1$ have the same leading coefficient α_1. We construct a sequence of operators $L_i' \in \hat{A}_i$, $i \geqslant 2$, with leading coefficient equal to α_1, such that $\operatorname{ord}(L_i' - L_{i+1}') \leqslant r - 2$. If the condition $\operatorname{ord}(L_i - L_{i+1}) \leqslant r - 2$ is satisfied for an infinite set of indices i, then we choose for the $\{L_i'\}$ a subsequence of the sequence $\{L_i\}$. In the opposite case, by changing to a subsequence we may assume that $\operatorname{ord}(L_{i+1} - L_i) = r - 1$. Then the operators $\tilde{L}_i = L_{i+2} - L_{i+1} \in \hat{A}_i$, and therefore their leading coefficients are proportional. It is easy to verify that with a suitable choice of c_i, the first two coefficients in the operator $L_i' = L_i + c_i \tilde{L}_i$, $i \geqslant 3$ coincide with the leading coefficients α_1, α_2 of the operator $L_2' \overset{\text{def}}{=} L_2$.

Continuing this process we arrive at a sequence of operators $M_1 = L_1$, $M_2 = L_2'$, ..., satisfying the conditions

$$M_k \in \hat{A}_k, \qquad \operatorname{ord}(M_{k+1} - M_k) \leqslant r - k.$$

The remaining part of this section is devoted to a discussion of the conditions for solvability of Eq. (1.8). These conditions are formulated in terms of residues:

$$\operatorname{res}\left(\sum_{i=-\infty}^{n} a_i D^i \right) \overset{\text{def}}{=} a_{-1}$$

of operators $\hat{A}_j$.

LEMMA 1.8 Suppose that $M, N \in \mathcal{F}((D^{-1}))$. Then $\operatorname{res}[M, N] \in \operatorname{Im} D$.

Proof It is clearly sufficient to prove the assertion for monomials $M = aD^k$, $N = bD^l$. For $k + l < -1$ we find that $\operatorname{res}[M, N] = 0$. For $k + l \geqslant -1$, we have

$$\operatorname{res}\left[aD^k, bD^l\right] = \frac{k(k-1)\cdots(1-l)(-l)}{(k+l+1)!} \times \left\{aD^{k+l+1}(b) + (-1)^{k+l} bD^{k+l+1}(a)\right\} \in \operatorname{Im} D.$$

Because of Lemma 1.6 the operator $F_* \in \hat{A}_{m-1}$ and, consequently, Eq. (1.8) is always solvable in $\hat{A}_{m-1}$.

THEOREM 1.9 Suppose that L is an arbitrary operator of first order in $\hat{A}_{m+k}$, $k \geqslant -1$. Then the condition

$$\begin{aligned} \partial_F(\operatorname{res} L^k) &\in \operatorname{Im} D, \qquad k \neq 0; \\ \operatorname{res}\left(L^{-1}\partial_F L\right) &\in \operatorname{Im} D, \qquad k = 0 \end{aligned} \tag{1.10}$$

is necessary and sufficient for the solvability of Eq. (1.8) in $\tilde{A}_{m+k-1}$.

Proof Suppose that Eq. (1.8) is solvable in $\hat{A}_{m+k+1}$. Then there exists an operator of first order $\tilde{L} \in \tilde{A}_{m+k+1}$, whose first $m + k$ coefficients coincide with the coefficients of the operator $L \in \hat{A}_{m+k}$. In fact, for $L_1 \in \hat{A}_{m+k+1}$, $\operatorname{ord} L_1 = r$, the operator L admits the representation (1.9):

$$L = \sum_{2-m-k}^{1} c_i L_1^{i/r} + bD^{1-m-k} + \cdots$$

and we can choose for $\tilde{L}$ the expression $\sum c_i L_i^{i/r} \in \hat{A}_{m+k+1}$. The operator $\tilde{L}$ satisfies the relations

$$\operatorname{res} \tilde{L}^k = \operatorname{res} L^k, \qquad \operatorname{ord}\left[\partial_F - F_*, \tilde{L}^k\right] \leqslant -2$$

Therefore

$$\partial_F(\operatorname{res} L^k) = \partial_F(\operatorname{res} \tilde{L}^k) = \operatorname{res}\left[F_*, \tilde{L}^k \right].$$

From this and Lemma 1.8 it follows that the condition (1.10) is satisfied for $k \neq 0$. For $k = 0$ the condition (1.10) follows from the relations

$$\operatorname{res}\left(L^{-1}\partial_F L\right) = \operatorname{res}\left(\tilde{L}^{-1}\partial_F \tilde{L}\right) = \operatorname{res}\left[\tilde{L}^{-1}, F_* \tilde{L} \right]$$

Suppose that condition (1.10) is satisfied and $k \neq 0$. Let us show that there exists an operator

$$M = a_k D^k + \cdots + a_0 + \cdots + a_{1-m} D^{1-m} + \alpha D^{-m}$$

belonging to $\hat{A}_{m+k+1}$. We set the coefficients $a_k, \ldots, a_{1-m}$ equal to the corresponding coefficients of the operator L^k. Then $M \in \hat{A}_{m+k}$ and the condition $M \in \hat{A}_{m+k+1}$ is equivalent to the condition

$$\partial_F(\operatorname{res} M) = \operatorname{res}\left[F_*, M \right]$$

Since $\operatorname{res} M = \operatorname{res} L^k$, the left side of this equation belongs to $\operatorname{Im} D$. The right side, according to Lemma 1.8, can be rewritten in the form

$$\operatorname{res}\left[\frac{\partial F}{\partial u_m} D^m, \alpha D^{-m} \right] + DH = mD\left(\alpha \frac{\partial F}{\partial u_m} \right) + DH$$

where H is a differential polynomial in the first $m + k$ coefficients of the operator L^k. Thus the last coefficient α of the operator M is calculated using the formula

$$\alpha = \left(m \frac{\partial F}{\partial u_m} \right)^{-1} \left(D^{-1} \partial_F \operatorname{res} L^k - H \right)$$

The case of $k = 0$ is treated similarly.

COROLLARY OF THEOREM 1.9 Equation (1.8) is solvable in $\hat{A}_{2m-2}$ if and only if the conditions

$$\partial_F \operatorname{res}\left(F_*^{k/m}\right) \in \operatorname{Im} D, \qquad -1 \leqslant k \leqslant m-3, \quad k \neq 0 \tag{1.11}$$

$$\partial_F\left(\frac{\partial F}{\partial u_{m-1}} \Big/ \frac{\partial F}{\partial u_m}\right) \in \operatorname{Im} D, \qquad m \geqslant 3 \tag{1.12}$$

are satisfied.

Proof The operator $L = F_*^{1/m} \in \hat{A}_{m-1}$, and the first of conditions (1.11) guarantees, because of Theorem 1.9, the solvability of (1.8) in $\hat{A}_m$. For $m = 2$ we have $2m-2 = m$ and the corollary is proved. For $m \geqslant 3$ we consider (cf. the proof of Theorem 1.9) the first order operator $\tilde{L} \stackrel{\text{def}}{=} L_1 \in \hat{A}_m$, such that $\operatorname{ord}(L_1 - L) < 3 - m$. The first $m-1$ coefficients of this operator L_1 are the same as for the operator $F_*^{1/m}$, and condition (1.10) for $k = 0$, as is easily checked, coincides for L_1 with condition (1.12). If condition (1.12) is satisfied and $m \geqslant 4$, we replace the operator L_1 by the first order operator $L_2 \in \hat{A}_{m+1}$ such that $\operatorname{ord}(L_2 - L_1) < 2 - m$, etc. It remains to be shown that, because of (1.10), the condition for solvability of Eq. (1.8) in $\hat{A}_{m+k+1}$ is formulated in terms of the first $k+2$ coefficients of the operator $L_i \in \hat{A}_{m+k}$. Since the first $m-1$ coefficients of the operator L_i, $i = 1, 2, \ldots$, are equal to the corresponding coefficients of the operator $F_*^{1/m}$, conditions (1.10) and (1.11) coincide for $k \leqslant m-3$, and from the fulfillment of conditions (1.11) there follows the solvability of (1.8) in $\hat{A}_{m+k+1}$, $m+k+1 \leqslant m+m-3+1 = 2m-2$.

The conditions (1.10) formulated above for the solvability of Eq. (1.8) have the form of conservation laws $\partial_F p_k = Dq_k$, $k = -1, 0, 1, 2, 3, \ldots$. According to the Corollary of Theorem 1.9, the densities $p_{-1}, p_0, p_1, \ldots, p_{m-3}$ are expressed explicitly in terms of the function F. For $k > m-3$ the density p_k is, generally speaking, expressed in terms of $p_{-1}, \ldots, p_{k-1}; q_{-1}, \ldots, q_{k-1}$, and the function F. For example, for $m = 2$

$$p_{-1} = \left(\frac{\partial F}{\partial u_2}\right)^{-1/2}, \qquad p_0 = \frac{\partial F}{\partial u_1} \Big/ \frac{\partial F}{\partial u_2} - p_{-1} q_{-1} \tag{1.12'}$$

1.3 Conservation laws

A conservation law $\partial_F p = Dq$ with a density $p \in \mathscr{F}$ is said to be trivial if $p \in \operatorname{Im} D + \mathbb{C}$, or, what is the same thing,

$$\frac{\delta p}{\delta u} \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (-1)^k D^k \frac{\partial p}{\partial u_k} = 0.$$

Let us find the necessary conditions for the existence of an infinite series of nontrivial conservation laws for Eq. (0.1). Applying the operation $*$ to both sides of the equality $\partial_F p = Dq$, we find that $p_*(F))_* = D \circ q_*$. Then $(p_*(F))'_* = -q'_* D$, where

$$\left(\sum a_k D^k\right)' \stackrel{\text{def}}{=} \sum (-1)^k D^k \circ a_k$$

It is not difficult to show that the coefficient of D^0 in the operator $(p_*(F))'_*$ can be written in the form $(\partial_F + F'_*)(\delta p/\delta u)$. Equating this expression to zero, we arrive at the well-known (cf. [12]) equation

$$\left(\partial_F + F'_*\right) g = 0 \tag{1.13}$$

for the variational derivative g of the density in the conservation law of Eq. (0.1).

The operation $*$ changes Eq. (1.13) to the equation

$$F'_* g_* + g_* F_* + \partial_F(g_*) + \sum_{k=0}^{m} \left(D_g^k\right)(G_k)_* = 0 \tag{1.14}$$

where $G_k = G_k(u, \ldots, u_{2m})$ are the coefficients of the differential operator F'_*. The last term in (1.14) is an operator of order no higher than $2m$. Therefore, for ord $g = n \geqslant m + 1$, the equations for determining the leading coefficients of the operator g_* coincide with the equations for the coefficients of the operator $s \in \mathscr{F}((D^{-1}))$ of order n, satisfying the relation

$$\partial_F(S) + F'_* S + S F_* = 0 \tag{1.15}$$

In constrast to Eq. (1.8), Eq. (1.15) cannot in principle have a solution in the case of even order m of Eq. (0.1). In fact, for even m

$$\operatorname{ord}\left(F_*^t S + S F_*\right) = m + n > \operatorname{ord} \partial_F(S)$$

It then also follows that for even m Eq. (1.13) has no solutions of order higher than m (cf. [13]).

From Lemma 1.4 it follows that if Eq. (1.8) has at least one solution, then it has a solution of any order. Equation (1.15) does not have this property. However, from the existence of two solutions of different order it follows, as we shall show below, that there exist solutions of any order.

LEMMA 1.10 Suppose that $S_1, S_2 \in \mathscr{F}((D^{-1}))$ are solutions of Eq. (1.15) of orders n_1, n_2 $(n_2 > n_1)$. Then $L = S_1^{-1} S_2$ is a solution of Eq. (1.8), and the general solution S of Eq. (1.15) is written in the form

$$S = S_1 \sum_{i=-\infty}^{n} c_i L^{\frac{i}{n_2 - n_1}}, \qquad c_i \in \mathbb{C}$$

Proof We have

$$\begin{aligned}
\left[\partial_F, S_1^{-1} S_2\right] &= S_1^{-1}\left[\partial_F, S_2\right] - S_1^{-1}\left[\partial_F, S_1\right] S_1^{-1} S_2 \\
&= -S_1^{-1}\left(F_*^t S_2 + S_2 F_*\right) + S_1^{-1}\left(F_*^t S_1 + S_1 F_*\right) S_1^{-1} S_2 \\
&= \left[F_*, S_1^{-1} S_2\right]
\end{aligned}$$

Similarly one verifies that $S_1 L_1$ is a solution of (1.15) if L_1 is an arbitrary solution of (1.8). The formula for the general solution S follows from Lemma 1.4.

To underline the analogy with Theorem 1.7, we denote by $B(F)$ ($\hat{B}(F)$) the set of solutions in $\mathscr{F}$ ($\mathscr{F}((D^{-1}))$) of Eqs. (1.13) and (1.15), respectively. We denote by $\hat{B}_k$ the set of operators $S \in \mathscr{F}((D^{-1}))$ satisfying the condition

$$\operatorname{ord}\left(\partial_F(S) + F_*^t S + S F_*\right) \leqslant \operatorname{ord}(S) + m - 1 - k$$

We shall say that Eq. (1.15) is solvable in $\hat{B}_k$ if $\hat{B}_k$ contains at least one nonzero element. We recall that from the solvability of Eq. (1.8) in $\hat{A}_k$ for all $k \in \mathbb{N}$ there follows (cf. the proof of Theorem 1.7) its solvability in $\hat{A}_\infty = \hat{A}(F)$.

THEOREM 1.12 Suppose that $B(F)$ contains elements of arbitrarily high order. Then Eqs. (1.8), (1.15) are solvable in $\hat{A}_k, \hat{B}_k$ for all $k \in \mathbb{N}$, and there exist operators of first order $L \in \hat{A}(F)$, $S \in \hat{B}(F)$ such that

$$S' = -S, \qquad L' = -SLS^{-1}$$

Abbreviated Proof Suppose that g_1, g_2 are solutions of orders n_1, n_2 $(n_2 > n_1)$ of Eq. (1.13). From (1.14) we find that $S_i \overset{\text{def}}{=} (g_i)_* \in \hat{B}_{n_i - m - 1}$, $i = 1, 2$. Following the proof of Lemma 1.11 we verify that $\tilde{L} = S_1^{-1} \times S_2 \in \hat{A}_k$, $k = n_1 - m - 1$. Since the orders n_2 and n_1 can be chosen arbitrarily large, Eq. (1.8) is solvable in $\hat{A}_k$ for any k and, consequently, there exists an operator of first order $L_1 \in \hat{A}(F)$.

The element $g \in B(F)$ of order n generates an operator of first order $g_* L_1^{1-n} \in \hat{B}_{n-m-1}$. Since n can be chosen arbitrarily large, we can apply to the sequence of operators thus obtained in $\hat{B}_k$, $k \to \infty$, the arguments used in the proof of Theorem 1.7. Thus from the solvability of (1.15) in $\hat{B}_k$ for any k there follows the existence of a first order operator in $\hat{B}(F)$.

Together with the solution S of order n, another solution of Eq. (1.15) is the operator $S' + (-1)^n S$ of order n, and the operator SL_1 of order $n + 1$. Thus Eq. (1.15) has a solution of first order S_1 and a solution of second order S_2, such that $S_1' = -S_1$, $S_2' = S_2$. It is easy to verify that the operators $S = S_1$ and $L = S_1^{-1} S_2$ are connected by the relation $L' = -SLS^{-1}$.

Suppose that the order m of Eq. (0.1) is odd ($m \geqslant 3$). In this case, relation (1.15) is equivalent to the chain of equations

$$m \frac{\partial F}{\partial u_m} D(a_k) + \left((m-k) D\left(\frac{\partial F}{\partial u_m}\right) - 2 \frac{\partial F}{\partial u_{m-1}}\right) a_k$$

$$= \overset{(n)}{R}_k(a_{k+1}, \ldots, a_n, F), \qquad k = n, n-1, \ldots \tag{1.16}$$

for determining the coefficients of the operator $S = \sum_{-\infty}^{n} a_i D^i$ (cf. (1.6)). It is easy to check that the first of the equations of the system

(1.16) has the form

$$m\frac{\partial F}{\partial u_m}D(a_n)+\left((m-n)D\left(\frac{\partial F}{\partial u_m}\right)-2\frac{\partial F}{\partial u_{m-1}}\right)a_n=0$$

In order that this equation have a solution $a_n \in \mathscr{F}$ it is necessary and sufficient that

$$\frac{\partial F}{\partial u_{m-1}}\Big/\frac{\partial F}{\partial u_m}\in \operatorname{Im} D \tag{1.17}$$

Relation (1.17) is a criterion for solvability of (1.15) in $\hat{B}_1$. Conditions for solvability of Eq. (1.15) in $\hat{B}_k$, $k \leqslant m-2$ will be obtained below. From (1.15),

$$\operatorname{ord}\left(F_*' + SF_*S^{-1}\right) \leqslant 0$$

Then, as is easily verified, it follows that

$$\operatorname{ord}\left\{\left(F_*^{k/m}\right)' + (-1)^{k-1}SF_*^{k/m}S^{-1}\right\} \leqslant k-m, \qquad k \in \mathbb{N}$$

Thus

$$\operatorname{res}\left(F_*^{k/m}\right)' = (-1)^k \operatorname{res} SF_*^{k/m}S^{-1}, \qquad k = 1,2,\ldots,m-2$$

and

$$\operatorname{res}\left\{\left(F_*^{k/m}\right)' + (-1)^{k-1}F_*^{k/m}\right\} = (-1)^k\operatorname{res}\left(SF_*^{k/m}S^{-1} - F_*^{k/m}\right)$$

$$= (-1)^k\operatorname{res}\left[S, F_*^{k/m}S^{-1}\right]$$

Therefore (cf. Lemma 1.8) for the solvability of (1.15) in $\hat{B}_k$, $k > 1$, it follows that

$$\operatorname{res}\left\{\left(F_*^{k/m}\right)' + (-1)^{k-1}F_*^{k/m}\right\} \in \operatorname{Im} D, \qquad k = 1,2,\ldots,m-2$$

For odd k these conditions are satisfied automatically, since $\operatorname{res} M = -\operatorname{res} M'$ for $M \in \mathscr{F}((D^{-1}))$. For even k they are written in the form

$$\operatorname{res} F_*^{k/m} \in \operatorname{Im} D, \qquad k = 2, 4, \ldots, m-3 \tag{1.18}$$

One can show that the conditions (1.17), (1.18) are not only necessary, but also sufficient for the solvability of (1.15) in $\hat{B}_{m-2}$. We further note that for $\partial F/\partial u_m = \text{const}$, an additional relation of the form (1.18) with $k = m - 1$ is the condition for solvability of (1.15) in $\hat{B}_m$.

Relations (1.17), (1.18) show that solvability of Eq. (1.15) imposes on the right side of Eq. (0.1) much stricter conditions than condition (1.11) for the solvability of Eq. (1.8). In particular, from (1.17), (1.18) it follows that the even-numbered conservation laws enumerated in the Corollary to Theorem 1.7 are trivial in the case of solvability of Eq. (1.15). In the Theorem given just below, the conditions for solvability of Eq. (1.15) are formulated in a form analogous to (1.18). The proof of Theorem 1.13 will be given after the proof of Theorem 1.16.

THEOREM 1.13 Suppose that the conditions of Theorem 1.2 are satisfied. Then there exists a first order operator $L \in \hat{A}(F)$ such that

$$\operatorname{res} L^{2k} \in \operatorname{Im} D, \qquad k \in \mathbb{N} \tag{1.19}$$

In contrast to the conditions (1.11) of Theorem 1.9, the form of the relation (1.19) depends on the choice of $L \in \hat{A}(F)$. For a first order operator L of the general form (cf. Lemma 1.4) relation (1.19) is replaced by the condition: the residue $\operatorname{res} L^{2k}$ for any $k \in \mathbb{N}$ is a linear combination of residues of odd powers of L modulo $\operatorname{Im} D$.

1.4 Conditions for formal linearizability

In this Section we discuss several variants for strengthening conditions (1.10) for solvability of Eq. (1.8). One of these variants (conditions (1.17), (1.19)) was mentioned in the preceding Section.

Definition 1.14 The operators $M, N \in \mathscr{F}((D^{-1}))$ are said to be equivalent ($M \sim N$) if there exists an operator $T \in \mathscr{F}((D^{-1}))$ such that

$$TMT^{-1} = N \tag{1.20}$$

LEMMA 1.15 For given M, N, the general solution T of Eq. (1.20) is given by the formula

$$T = T_1 \sum_{i=-\infty}^{n} c_i M^{i/r}, \qquad c_i \in \mathbb{C} \tag{1.21}$$

where T_1 is one of the solutions of (1.20), $r = \operatorname{ord} M$.

Proof The operator T defined by formula (1.21) satisfies relation (1.20). Suppose that $\tilde{T}$ is any solution of (1.20). Then $T_1^{-1}\tilde{T}$ commutes with M. Therefore (cf., for example, [11]) the operator $T_1^{-1}\tilde{T}$ can be represented in the form $\sum c_i M^{i/r}$.

It is clear that equivalent operators have the same order and identical leading coefficients. Furthermore, equivalence of M and N is tantamount to equivalence of $M^{1/r}$ and $N^{1/r}$, $r = \operatorname{ord} M = \operatorname{ord} N$. Therefore, without loss of generality we may assume that in relation (1.20)

$$\begin{aligned} M &= aD + a_0 + a_{-1}D^{-1} + \cdots, \\ N &= aD + b_0 + b_{-1}D^{-1} + \cdots. \end{aligned} \tag{1.22}$$

By virtue of Lemma 1.15 one may also assume that $\operatorname{ord} T = 1$.

THEOREM 1.16 The operators (1.22) are equivalent if and only if

$$a^{-1}(a_0 - b_0) \in \operatorname{Im} D \tag{1.23}$$

and

$$\operatorname{res}(N^k - M^k) \in \operatorname{Im} D, \qquad k \in \mathbb{N} \tag{1.24}$$

Proof Suppose that $T = \alpha_1 D + \alpha_0 + \alpha_{-1}D^{-1} + \cdots$. Comparing coefficients of D^0 in (1.20) we get an equation for determining α_1:

$$\alpha_1^{-1}D\alpha_1 - a^{-1}Da = a^{-1}(a_0 - b_0)$$

The condition for solvability of this equation in $\mathscr{F}$ is relation (1.23). Since the conditions for solvability are unchanged if we replace M, N by M^k, N^k, the following equations for determining $\alpha_0, \alpha_{-1}, \ldots$ can be obtained by equating coefficients of D^{-1} in the relations

$$TM^kT^{-1} = N^k, \qquad k = 1, 2, \ldots$$

We rewrite the equality $\operatorname{res}(TM^kT^{-1} - N^k) = 0$ in the form

$$\operatorname{res}(N^k - M^k) = \operatorname{res}[TM^k, T^{-1}].$$

One can verify that (cf. the proof of Theorem 1.9) in order for the coefficient $\alpha_{1-k} \in \mathscr{F}$ to be determined from this relation it is necessary and sufficient that $\operatorname{res}(N^k - M^k) \in \operatorname{Im} D$.

Theorem 1.13 of the preceding Section is an immediate consequence of Theorems 1.12 and 1.13. In fact, because of Theorem 1.12 there exists a first order operator $L \in \hat{A}(F)$, satisfying the condition $L' \sim -L$. From formula (1.24), for $M = L$, $N = -L'$, we get

$$\operatorname{res}\{L^n - (-1)^n(L^n)'\} \in \operatorname{Im} D$$

Then (1.19) follows for $n = 2k$.

Definition 1.17 Equation (0.1) is said to be formally linearizable if the densities p_n in the conservation laws (1.10) satisfy the conditions

$$p_n \in \operatorname{Im} D + \mathbb{C}, \qquad n = -1, 0, 1, 2, \ldots \tag{1.25}$$

We recall that, because of (1.10)–(1.12),

$$p_{-1} = \left(\frac{\partial F}{\partial u_m}\right)^{-1/m},$$

$$p_0 = \frac{\partial F}{\partial u_{m-1}} \Big/ \frac{\partial F}{\partial u_m} - \begin{cases} 0, & m \geqslant 3 \\ p_{-1} D^{-1} \partial_F p_{-1}, & m = 2 \end{cases} \tag{1.26}$$

$$p_n = \operatorname{res} L^n, \qquad n = 1, 2, 3, \ldots \tag{1.27}$$

From Lemma 1.4 it follows that the conditions (1.25) do not change their form when we replace one first-order operator $L \in \hat{A}(F)$ by another. Moreover, because of Theorem 1.9, for any $k \in \mathbb{N}$, condition (1.25) with $n = k$ is independent of the conditions with $n > k$, and is formulated in terms of the first $k + 1$ densities $p_{-1}, p_0, \ldots, p_{k-1}$ in the conservation laws (1.10) (cf. the Corollary to Theorem 1.9).

THEOREM 1.18 Suppose that $\partial F/\partial u_m = \text{const}$. Then the following assertions are equivalent:

1) Equation (0.1) is formally linearizable;
2) There exists in $\hat{A}(F)$ an operator $L \sim D$;
3) There exists an operator $T \in \mathscr{F}((D^{-1}))$ such that

$$T(\partial_F - F_*)T^{-1} = \partial_F - \sum_{-\infty}^{m} c_k D^k, \qquad c_k \in \mathbb{C} \tag{1.28}$$

Proof The equivalence 1)$\Leftrightarrow$2) follows from Theorem 1.16. From (1.28) we find that the operator $L = T^{-1}DT$ commutes with $\partial_F - F_*$ and, therefore, 3)$\Rightarrow$2). The converse is also true since, from the relations,

$$L = T^{-1}DT, \qquad [L, \partial_F - F_*] = 0$$

it follows that the operator $T(\partial_F - F_*)T^{-1}$ commutes with D, and, consequently, satisfies the relation (1.28).

In similar fashion one proves

THEOREM 1.19 The following assertions are equivalent:

1) Eq. (0.1) is formally linearizable;
2) There exists in $\hat{A}(F)$ an operator $L \sim D' \overset{\text{det}}{=} (\partial F/\partial u_m)^{1/m} D$;
3) There exist $T \in \mathscr{F}((D^{-1}))$, $\gamma \in \mathscr{F}$, such that for $\partial'_F \overset{\text{def}}{=} \partial_F + \gamma D'$, the relations

$$T(\partial_F - F_*)T^{-1} = \partial'_F - \sum_{-\infty}^{m} c_i (D')^i, \qquad c_i \in \mathbb{C} \tag{1.29}$$

$$[\partial'_F, D'] = 0 \tag{1.30}$$

are satisfied.

We note as a supplement to Theorem 1.19 that from (1.30) it follows that the first of conditions (1.25) is satisfied, and also the formula $\gamma = -D^{-1}(\partial_F p_{-1})$.

The standard example for Theorem 1.18 is the class of equations (0.1), which are reduced to linear equations with constant coefficients by the differential substitution

$$u' = \varphi(u, u_1, \ldots, u_n) \tag{1.31}$$

For equations of this class the operator T of condition 3) is differential (cf., for example, [2]) and coincides with φ_*. The form of the substitution (1.30) can be found by a direct computation analogous to that in the example presented below.

Example 1.20: Let us find a substitution $u' = \varphi(u, u_1)$ that reduces the equation

$$u_t = u_3 - \frac{3}{4}\frac{u_2^2}{u_1} \tag{1.32}$$

to linear form $u'_t = u'_3$. Substituting $u' = \varphi(u, u_1)$ in the equation $u'_t = u'_3$, we find

$$\varphi_0\left(u_3 - \frac{3}{4}\frac{u_2^2}{u_1}\right) + \varphi_1\left(u_4 - \frac{3}{2}\frac{u_2 u_3}{u_1} + \frac{3}{4}\frac{u_2^3}{u_1^2}\right)$$

$$= \varphi_1 u_4 + \varphi_0 u_3 + 3\varphi_{01}u_1u_3 + 3\varphi_{01}u_2^2 + 3\varphi_{00}u_1u_2 + 3\varphi_{11}u_2u_3$$

$$+ \varphi_{000}u_1^3 + 3\varphi_{001}u_1^2u_2 + 3\varphi_{011}u_1u_2^2 + \varphi_{111}u_2^3$$

where $\varphi_0 = \partial\varphi/\partial u$, $\varphi_1 = \partial\varphi/\partial u_1$, etc. Comparing terms containing u_3, we find $3\varphi_{01}u_1 + 3\varphi_{11}u_2 + \frac{3}{2}\varphi_1(u_2/u_1) = 0$. Since φ does not depend on u_2, we have $\varphi_{11} + \varphi_1/2u_1 = 0$, $\varphi_{01} = 0$. From this, $\varphi = cu_1^{1/2} + A(u)$, $c \in \mathbb{C}$. Further fixing of the form of the function φ gives the formula for the substitution

$$\varphi = cu_1^{1/2} + c_1, \qquad c_1 \in \mathbb{C}$$

It is clear that, without loss of generality, we may set $c = 1$, $c_1 = 0$.

A natural generalization of the Definition 1.17 leads to the case where almost all (i.e., except for a finite number) of the conservation laws (1.10) are trivial. In this case we can introduce a differential extension of the algebra $\mathcal{F}$, adding to the variable u new differential variables $\overset{(i)}{u} = D^{-1}p_i$, corresponding to the densities (1.26), (1.27) of the nontrivial conservation laws (1.10). The differential extension of the algebra $\mathscr{F}$ enables us to generalize formula (1.31), including among the arguments of the function φ not only $u, u_1, u_2, \ldots,$ but also the new differential variables $\overset{(i_1)}{u}, \ldots, \overset{(i_p)}{u}$. The procedure for calculating the explicit form of the substitution

$$u' = \varphi\left(\overset{(i_1)}{u}, \ldots, \overset{(i_p)}{u}, u, u_1, \ldots, u_n\right) \tag{1.33}$$

reducing the Eq. (0.1) to a linear one, is not essentially different, since $\partial_F \overset{(i)}{u} \in \mathscr{F}$, from the case of (1.31).

Example 1.21 For the equation

$$u_t = u_3 + u^2 u_2 + 3uu_1^2 + \tfrac{1}{3}u^4 u_1 \tag{1.34}$$

we have

$$p_0 = \frac{\partial F}{\partial u_2} \Big/ \frac{\partial F}{\partial u_3} = u^2 \notin \operatorname{Im} D + \mathbb{C}$$

Since $\partial_F p_0 \in \operatorname{Im} D$, we can introduce a differential extension $\mathscr{F}[v]$, $v = D^{-1}p_0 = D^{-1}u^2$. The substitution

$$u' = u \exp\left\{\tfrac{1}{3}D^{-1}u^2\right\}$$

relating Eq. (1.34) to the linear equation $u'_t = u'_3$, is found by direct calculation, as in the Example 1.20.

For any equation having a conservation law of zero order with density $p(u)$, the introduction of $v = D^{-1}p$ leads to an equation of the form $v_t = F(v, v_1, \ldots, v_m)$, $\partial F/\partial v = 0$. It is not difficult to verify that in Example 1.21 this equation for v is equivalent to the equation of Example 1.20 to within the substitution $v \leftrightarrow \psi(u)$.

At the conclusion of this Section we consider briefly the question of the interrelation of the algebras $A(F), \hat{A}(F)$ and $A(F'), \hat{A}(F')$ of the equations

$$u_t = F(u, u_1, \ldots, u_m), \qquad u'_t = F'(u', u'_1, \ldots, u'_m) \tag{1.35}$$

related by the differential substitution (1.31). We denote by $\mathscr{F}'$ the algebra of functions of the variables $u', u_1', u_2', \ldots,$ with the derivation $D' = \sum u_{i+1}' \partial/\partial u_i'$. From the point of view of differential algebra the substitution (1.31) is an injective homomorphism $\sigma: \mathscr{F}' \to \mathscr{F}$, satisfying the two conditions

$$\sigma D' = D\sigma, \qquad \partial_F(\operatorname{Im}\sigma) \subset \operatorname{Im}\sigma \tag{1.36}$$

Because of the first of conditions (1.36), the action of σ on any element of $\mathscr{F}'$ is determined by the function $\varphi(u, \ldots, u_n) \stackrel{\text{def}}{=} \sigma(u')$. The second condition guarantees the correctness of the definitions $\partial_{F'} \stackrel{\text{def}}{=} \sigma^{-1}\partial_F \sigma$, $F' \stackrel{\text{def}}{=} \partial_{F'}(u')$ of the evolutionary derivation $\partial_{F'}$ and the right side F', associated with the transformed equation (1.35).

Suppose that $g \in A(F)$, i.e., $[\partial_g, \partial_F] = 0$. It is clear that if $\partial_g(\operatorname{Im}\sigma) \subset \operatorname{Im}\sigma$ then $\sigma^{-1}\partial_g\sigma$ is an evolutionary derivation in $\mathscr{F}'$, commuting with $\partial_{F'}$. Thus, the subalgebra in $A(F)$, consisting of elements $g \in A(F)$ such that $\partial_g(\operatorname{Im}\sigma) \subset \operatorname{Im}\sigma$, is mapped isomorphically into $A(F')$.

Conversely, suppose that $g' \in A(F')$, $\delta \stackrel{\text{def}}{=} \sigma\delta_{g'}\sigma^{-1}$. It is clear that δ is a derivation in $\operatorname{Im}\sigma$, commuting with D. The question arises whether δ is always extendable to a derivation in $\mathscr{F}$. The Example 1.20 shows that the answer to this is negative. In fact, an arbitrary element $g' \in A(F')$ in this particular case has the form $g' = \sum c_i u_i'$. We now find

$$\delta\left(\sqrt{u_1}\right) = \sum c_i D^i\left(\sqrt{u_1}\right) \Rightarrow \delta(u_1) = 2\sqrt{u_1} \sum c_i D^i\left(\sqrt{u_1}\right).$$

Therefore, for the existence in $\mathscr{F}$ of an evolutionary derivation ∂, such that $\partial|_{\operatorname{Im}\sigma} = \delta$, it is necessary and sufficient that the condition $\sqrt{u_1}\sum c_i D^i(\sqrt{u_1}) \in \operatorname{Im} D$ be fulfilled. It is not difficult to verify that this condition is satisfied if and only if $c_i = 0$ for even i.

Although the algebras $A(F')$ and $\hat{A}(F')$ are closely related, the difficulties with the definition $\partial_g \stackrel{\text{def}}{=} \sigma\,\partial_{g'}\sigma^{-1}$ disappear if we go over to the operator language.

LEMMA 1.22 Suppose that conditions (1.36) are satisfied. Then, from the solvability of the equation $[\partial_{F'} - F_*', L] = 0$ in $\mathscr{F}((D^{-1}))$, there follows the solvability of the equation $[\partial_F - F_*, L] = 0$ in $\mathcal{F}((D^{-1}))$.

Proof From the formulas $\varphi = \sigma(u')$, $F' = \partial_{F'}(u')$ it follows that $\varphi_*(F) = \sigma(F')$. Applying the operation $*$ to this relation, and using the fact that $(\sigma F')_* = \sigma F'_* \sigma^{-1}\varphi_*$, we get

$$\sigma(\partial_{F'} - F'_*)\sigma^{-1} = \varphi_*(\partial_F - F_*)\varphi^{-1}_* \tag{1.37}$$

We note that for any operator of $\mathcal{F}'((D'^{-1}))$,

$$\sigma\left(\sum a_i(D')^i\right)\sigma^{-1} \stackrel{\text{def}}{=} \sum \sigma(a_i)D^i,$$

Setting $L = \varphi_*^{-1}(\sigma L'\sigma^{-1})\varphi_*$, from (1.37) we get the assertion of the Lemma.

Remark The assertion of Lemma 1.22 remains valid if we replace the question of the solvability of Eq. (1.8) by the question of solvability of Eq. (1.15). Then, by virtue of (1.37),

$$S = \varphi'_*\sigma S'\sigma^{-1}\varphi_*$$

In papers [14], [15], in connection with the equations $u_t = u^2u_2$, $u_t = u^3u_3$, have been considered the differential substitution

$$u' = \varphi(u, \ldots, u_p), \qquad x' = \psi(u, \ldots, u_q) \tag{1.38}$$

which reminds one of the Legendre transformation. From the algebraic point of view, to the substitution (1.38) there corresponds an injective homomorphism $\sigma : \mathcal{F}' \to \mathcal{F}$ with the properties

$$\sigma D' = \alpha D\sigma, \qquad (\partial_F + \gamma_\alpha D)(\operatorname{Im}\sigma) \subset \operatorname{Im}\sigma \tag{1.39}$$

generalizing (1.36). In (1.39) α is a given function in $\mathcal{F}$, while the function $\gamma \in \mathcal{F}$ is determined from the commutation relation

$$\left[\partial_F + \gamma\alpha D, \alpha D\right] = 0 \tag{1.40}$$

The criteria for solvability of (1.40) are formulated (cf. Theorem 1.19) in the form $\partial_F \alpha^{-1} \in \operatorname{Im} D$. It is not difficult to check that for a homomorphism σ satisfying the conditions (1.39), (1.40), there remains valid the assertion of Lemma 2.12 and formula (1.37), where

$$\varphi \overset{\text{def}}{=} \sigma(u'), \qquad \sigma\left(\sum a_i (D')^i\right)\sigma^{-1} \overset{\text{def}}{=} \sum \sigma(a_i)(\alpha D)^i.$$

§2 Equations of Second and Third Order

In this Section we shall enumerate evolution equations (0.1) of order $m = 2, 3$, satisfying a few of the first conditions (1.10) of Theorem 1.9. We verify that for these equations at least the next condition (1.10) in order is satisfied automatically. However, we do not possess at present a proof of the solvability of Eq. (1.8) for all the equations enumerated in this Section.

For $m = 2$ one has succeeded, due to the efforts of S. I. Svinolupov, in overcoming the difficulties associated with the investigation of the compatibility of the system of partial differential equations equivalent to the first three conditions (1.10) of a function F of general form. For $m = 3$, the analogous problem is solved for equations satisfying the condition $\partial F/\partial u_3 = \text{const}$.

Since the conditions for solvability are invariant under point transformations $u \leftrightarrow \varphi(u)$, the classification is carried out to within such transformations. A complete solution of the classification problem requires the investigation of the equivalence of the equations obtained from the point of view of transformations of the form (1.33), but this interesting question goes beyond the scope of the present survey.

The solvability conditions (1.10) to be considered have the form of conservation laws. Therefore, in the classification, together with making more precise the form of the equation $u_t = F$, we determine the form of its conservation laws of lowest orders. We recall that the order of a conservation law $p_t + q_x = 0$ is the lowest of the orders mutually equivalent densities. Functions $p, \tilde{p}$ are considered to be equivalent ($p \equiv \tilde{p}$) if their linear combination belongs to $\operatorname{Im} D + \mathbb{C}$.

2.1 Classification of equations $u_t = F(u, u_1, u_2)$

For $m = 2$ only the first of conditions (1.10) is explicit. We write it in the form

$$\alpha_t \in \operatorname{Im} D, \qquad \alpha \stackrel{\text{def}}{=} \left(\frac{\partial F}{\partial u_2} \right)^{-1/2} \tag{2.1}$$

The next three conservation laws (1.10), with indices $k = 0, 1, 2$, can be written (cf. (1.12′)), by replacing the densities p_k by equivalent ones, as follows:

$$(\alpha\beta)_t \in \operatorname{Im} D, \qquad \beta \stackrel{\text{def}}{=} -\alpha \frac{\partial F}{\partial u_1} + D^{-1}(\alpha_t) + D(\alpha^{-1}) \tag{2.2}$$

$$\gamma_t \in \operatorname{Im} D, \qquad \alpha^{-1}\gamma \stackrel{\text{def}}{=} \frac{\partial F}{\partial u_0} - \frac{\beta^2}{4} + \frac{\beta}{2} D^{-1}(\alpha_t) - \frac{1}{2} D^{-1}\left[(\alpha\beta)_t\right] \tag{2.3}$$

$$\gamma D^{-1}(\alpha_t) - \alpha D^{-1}(\gamma_t) \in \operatorname{Im} D \tag{2.4}$$

First let us consider the general question of conservation laws for the second order equations.

LEMMA 2.1 The order of the conservation laws for the equation $u_t = F(u, u_1, u_2)$ are 0 or 1. Equations having conservation laws of order zero are quasilinear, i.e., they satisfy the condition

$$\frac{\partial \alpha}{\partial u_2} = 0 \qquad \left(\alpha = \left(\frac{\partial F}{\partial u_2} \right)^{-1/2} \right)$$

For equations with conservation laws of first order,

$$\frac{\partial \alpha}{\partial u_2} \neq 0, \qquad \frac{\partial^2 \alpha}{\partial u_2^2} = 0$$

The proof of this, as well as the following (Lemmas 2.2, 2.3) assertions is elementary, and we give the proof only for Lemma 2.3.

From Lemma 2.1 and the condition (2.1) it follows that the function $\alpha(u, u_1, u_2)$ is linear in u_2. Indeed, if the conservation law (2.1) is nontrivial, the lemma is applicable; if it is trivial, then

$$\alpha(u, u_1, u_2) = Df(u, u_1) = u_2 \frac{\partial f}{\partial u_1} + u_1 \frac{\partial f}{\partial u}$$

For $\partial\alpha/\partial u_2 \neq 0$, the equation under consideration is written in the form

$$u_t = \left[A(u, u_1)u_2 + B(u, u_1) \right]^{-1} + C(u, u_1) \tag{2.5}$$

Its conservation laws have order 1 or are trivial. The equation with $\partial\alpha/\partial u_2 = 0$ is quasilinear and is conveniently written in the form

$$u_t = Df(u, u_1) + C(u, u_1)u_1 \tag{2.6}$$

Unlike (2.5), this equation cannot have conservation laws of first order.

We begin the classification of quasilinear equations with a sharpening of the result of Lemma 2.1.

LEMMA 2.2 The criterion for existence of nontrivial laws for equation (2.6) is $\partial g/\partial u_1 = 0$, where

$$g \stackrel{\text{def}}{=} \frac{\partial c}{\partial u_1} \Big/ \frac{\partial f}{\partial u_1}$$

For $g = g(u)$ the density in any conservation law is equivalent to the function

$$a(u) = \int^u \exp\left\{ \int^{u'} g(u'')\, du'' \right\} du'$$

Lemma 2.2 and condition (2.1) allow one to find the form of the function $\alpha(u, u_1) = (\partial f/\partial u_1)^{-1/2}$ for the quasilinear equations (2.6).

If the conservation law (2.1) is trivial, then

$$\alpha(u, u_1) = Dh(u) + \text{const}$$

For $h' = 0$ we get the equation

$$u_t = u_2 + C(u, u_1)u_1 \tag{2.7}$$

For $h' \neq 0$, after the point transformation $u \leftrightarrow h(u)$, we have $h(u) = u$, $\alpha = u_1 + \text{const}$. The corresponding equations have the form

$$u_t = \frac{u_2}{u_1^2} + C(u, u_1)u_1 \tag{2.8}$$

or

$$u_t = \frac{u_2}{(u_1 + 1)^2} + C(u, u_1)u_1$$

The equations enumerated exhaust, to within transformations $u \leftrightarrow h(u)$, the list of quasilinear equations for which the conservation law (2.1) is trivial.

If the conservation law is nontrivial, then, in the notation of Lemma 2.2, we have

$$\alpha(u, u_1) = a(u) + Dh(u)$$

For $h' = 0$, setting $a(u) = u^{-1}$, we get

$$u_t = u^2 u_2 = C(u, u_1)u_1$$

For $h' \neq 0$, setting $h(u) = u$, gives

$$u_t = \frac{u_2 + a' u_1}{(u_1 + a)^2} + C(u, u_1)u_1 \tag{2.9}$$

The form of the right side of these equations is determined from the condition (2.2) and Lemma 2.2. For example, for Eq. (2.8) we have

$$\alpha\beta = -u_1^3 \frac{\partial c}{\partial u_1} + 2D \ln u_1, \qquad \frac{\partial c}{\partial u^1} \Big/ \frac{\partial f}{\partial u_1} = u_1^2 \frac{\partial c}{\partial u_1}$$

If there is a nontrivial conservation law, then by Lemma 2.2,

$$u_1^2 \frac{\partial c}{\partial u_1} = g(u)$$

In the opposite case the conservation law (2.2) is trivial, i.e.,

$$u_1^3 \frac{\partial c}{\partial u_1} = h(u)u_1 + \lambda, \qquad \lambda = \text{const}$$

In the first case, $c = -u_1^{-1}g(u) + f(u)$, in the second

$$c = -\frac{h(u)}{u_1} - \frac{\lambda}{2u_1^2} + f(u)$$

In both cases the third and fourth conditions are satisfied automatically. For Eqs. (2.7) and (2.9) the third condition brings further sharpening. Below we give a list of quasilinear equations satisfying conditions (2.1)–(2.3):

$$u_t = u_2 + uu_1, \qquad a(u) = u; \tag{2.10}$$

$$u_t = u^2(u_2 + c_1u_1), \qquad a(u) = u^{-1}; \tag{2.11}$$

$$u_t = \frac{u_2}{u_1^2} - \frac{a''(u)}{a'(u)} + f(u)u_1; \tag{2.12}$$

$$u_t = \frac{u_2}{u_1^2} + \frac{1}{u_1} + f(u) + g(u)u_1; \tag{2.13}$$

$$u_t = \frac{u_2}{(u_1+1)^2} + \frac{[c_1 + f(u_1+1)]^2 - f'u_1^2}{(c_1+f)(u_1+1)} \tag{2.14}$$

$$u_t = \frac{u_2}{(u_1+1)^2} + \frac{a''}{a'(u_1+1)} - \frac{a''}{a'} + \left(\frac{a''}{a'} + c_1a\right)u_1 \tag{2.15}$$

$$u_t = \frac{u_2 + a'u_2}{(u_1+a)^2} + \frac{a''a}{a'(u_1+a)} - \frac{a''}{a'} + \left(\frac{a''}{a'a} - \frac{a'}{a^2} + \frac{c_1}{a^2}\right)u_1 \tag{2.16}$$

We have omitted the linear equation $u_t = u_2 + c_1 u + c_2$ from this list. Besides, we can add a term $c_2 u_1$ on the right side of the equations, without affecting the solvability of Eq. (0.2). Somewhat unexpectedly, the coefficients of Eqs. (2.12)–(2.16) include arbitrary functions $a(u)$, $f(u)$, $g(u)$. We recall that the group of point transformations has already been used to bring the equations to their simplest canonical form.

Equations (2.13), (2.14) have only trivial conservation laws. The coefficients of the remaining equations (2.10)–(2.12), (2.15)–(2.16) are written in terms of the density $a(u)$ of the nontrivial conservation law. All the equations having a conservation law (cf., the Example, 1.21) admit the substitution $v = D^{-1}a(u)$. The equation obtained as a result of this substitution, $v_t = F$, is invariant under the transformation $v \leftrightarrow v + \text{const}$, i.e., $F = F(v_1, v_2)$. After the substitution, Eqs. (2.12), (2.15), (2.16) become essentially nonlinear, and are written in the form:

$$v_t = \frac{-1}{f^2(v_1)v_2} + g(v_1) \tag{2.12'}$$

$$v_t = \frac{-1}{f(v_1)(f(v_1)v_2 + 1)} + \frac{1}{f(v_1)} + c_1 v_1^2 \tag{2.15'}$$

$$v_t = \frac{-1}{f(v_1)(f(v_1)v_2 + v_1)} + \frac{1}{f(v_1)v_1} + \frac{c_1}{v_1} \tag{2.16'}$$

Here $f = f(v_1) = 1/a'(u)$ with $v_1 = a(u)$.

For essentially nonlinear equations of the form (2.5) the connection of conditions (2.2)–(2.4) with the conditions for the existence of nontrivial conservation laws is established by means of the analog of Lemma 2.2.

LEMMA 2.3 For Eq. (2.5) the criterion for the existence of nontrivial conservation laws is the relation

$$\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial u_1}\frac{B}{Au_1}\right)g = \left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial}{\partial u_1}u_1^{-1}\right)\frac{B}{Au_1} \tag{2.17}$$

where

$$g \stackrel{\text{def}}{=} u_1^{-1} + \frac{\partial \ln A}{\partial u_1} + A\left(\frac{B}{Au_1} + u_1\frac{\partial}{\partial u} - \frac{B}{Au_1}\frac{\partial}{\partial u_1}\right)C \tag{2.18}$$

When condition (2.17) is satisfied the density of the conservation law is calculated from the formula

$$a(u,u_1) = \int_0^{u_1} du_1' \int_0^{u_1'} du_1'' \exp\left\{\int_0^{u_1''} g(u,u_1''')\,du_1'''\right\} \tag{2.19}$$

Proof Let $\partial_F a(u,u_1) \in \operatorname{Im} D$. Then

$$(Au_2 + B)^{-1}\left(\frac{\partial a}{\partial u} - D\left(\frac{\partial a}{\partial u_1}\right)\right) + C\left(\frac{\partial a}{\partial u} - D\left(\frac{\partial a}{\partial u_1}\right)\right) = Df(u,u_1)$$

or

$$\frac{B}{Au_1}\frac{\partial^2 a}{\partial u_1^2} = \frac{\partial}{\partial u}\left(\frac{\partial a}{\partial u_1} - u_1^{-1}a\right),$$

$$\frac{\partial}{\partial u}\left(C\frac{\partial^2 a}{\partial u_1^2}\right) = \frac{\partial}{\partial u_1}\left[\frac{1}{Au_1}\frac{\partial^2 a}{\partial u_1^2} + C\frac{\partial}{\partial u}\left(\frac{\partial a}{\partial u_1} - \frac{a}{u_1}\right)\right]$$

This system of equations for determining $a(u,u_1)$ is equivalent to the system

$$\frac{B}{Au_1}\frac{\partial^2 a}{\partial u_1^2} = \frac{\partial}{\partial u}\left(\frac{\partial a}{\partial u_1} - \frac{a}{u_1}\right), \qquad \frac{\partial}{\partial u_1}\ln\frac{\partial^2 a}{\partial u_1^2} = g \tag{2.20}$$

Condition (2.17) is the compatibility condition of the system (2.20). When it is satisfied the general solution of (2.20) is written in the form

$$a(u,u_1) = c_1\tilde{a} + Dh(u) + c_2, \qquad c_1, c_2 \in \mathbb{C},$$

where we denote by $\tilde{a}$ the right side of formula (2.19).

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Lemma 2.3 allows us to construct equations satisfying the condition (2.1), with an arbitrary function $a(u, u_1)$ as the density in the conservation law. The coefficients of this equation of the form (2.5) are expressed in terms of the function $a(u, u_1)$ by means of formulas (2.20), (2.18) and the condition (2.1). The last, because of the uniqueness of the conservation law, gives

$$\alpha = \left(\frac{\partial F}{\partial u_2}\right)^{-1/2} = \frac{Au_2 + B}{\sqrt{-A}} = Df(u, u_1) + \delta \cdot a(u, u_1) + \lambda(1 - \delta)$$

where $\delta = 0, 1$, $\lambda \in \mathbb{C}$. As the equation for determining the function f, and also the coefficients A, B, we have the first of the relations (2.20):

$$a_{11}u_1 \frac{\partial f}{\partial u} + (a_0 - a_{10}u_1)\frac{\partial f}{\partial u_1} + \left[\delta a + \lambda(1 - \delta)\right]a_{11} = 0 \qquad (2.20')$$

Here and in the sequel $a_0 = \partial a/\partial u$, $a_{10} = \partial^2 a/\partial u\,\partial u_1$, $a_{11} = \partial^2 a/\partial u_1^2$. Assuming that the function f is known from (2.21) (this equation is integrated by quadratures), we can find the last coefficient $C(u, u_1)$ by solving Eq. (2.18), whose left side is connected with the density $a(u, u_1)$, by the second of formulas (2.20). Thus, for $\delta = \lambda = 0$ we get the equation

$$\begin{gathered} u_t = -(f_1 Df)^{-1} + C(u, u_1) \\ \left[a_{10} - \frac{a_0}{u_1} + a_{11}u_1\frac{\partial}{\partial u} - (a_{10}u_1 - a_0)\frac{\partial}{\partial u_1}\right]C(u, u_1) \\ = \frac{2a_{11}f_{11}}{f_1^3} + \frac{a_{11}}{f_1^2 u_1} - \frac{a_{111}}{f_1^2} \end{gathered} \qquad (2.21)$$

For any function $a(u, u_1)$, $\partial a/\partial u_1 = 0$, this equation satisfies, surprisingly, not only the condition (2.1), but also the conditions (2.2), (2.3). Condition (2.2) is equivalent to (2.17), while condition (2.3) is satisfied automatically.

For $\delta = 0$, $\lambda \neq 0$, condition (2.2) is stronger than (2.17), which enables us to sharpen the form of the solution $C(u, u_1)$ of Eq. (2.18).

This gives

$$u_t = -\frac{1}{f_1(f_1u_2 + f_0u_1 + \lambda)} + C(u, u_1)$$
$$C = \lambda^{-1}\left(\frac{a_{111}u_1}{a_{11}f_1} - \frac{f_{11}u_1}{f_1^2} + \frac{1}{f_1}\right) + \alpha(a_1u_1 - a)u_1, \qquad \alpha \in \mathbb{C} \tag{2.22}$$

Equation (2.22) automatically satisfies condition (2.3).

The case of $\delta = 1$ is analogous to the preceding one. We have

$$u_t = -f_1^{-1}(Df + a)^{-1} + C(u, u_1)$$
$$C(u, u_1) = -\frac{a_{111}u_1}{a_{11}(a_1u_1 - a)f_1} + \frac{a_{11}u_1^2 + \mu f_1u_1}{f_1(a_1u_1 - a)^2} + \frac{f_{11}u_1 - f_1}{f_1(a_1u_1 - a)} \tag{2.23}$$

Condition (2.2) is used to sharpen the form of the solution of Eq. (2.18). Condition (2.3) is satisfied automatically.

Equations (2.21)–(2.23) exhaust the list of Eqs. (2.5) with nontrivial conservation law, satisfying the conditions (2.1)–(2.3). There remains to enumerate equations without conservation laws, for which the conditions (2.1)–(2.3) are satisfied in a stronger formulation. Condition (2.1) goes over into the condition

$$\alpha = \frac{Au_2 + B}{\sqrt{-A}} = Df(u, u_1) + \lambda, \qquad \lambda \in \mathbb{C}$$

The next condition $\alpha\beta \in \operatorname{Im} D + \mathbb{C}$ leads to an overdetermined system of equations for determining the coefficient $C(u, u_1)$ of Eq. (2.5). The condition for compatibility of this system is expressed in the form of an equation for determining the function $g(u, u_1)$:

$$\left[\frac{\lambda}{u_1} + f_1u_1\frac{\partial}{\partial u} - (f_0u_1 + \lambda)\frac{\partial}{\partial u_1}\right]g + 2\frac{f_{01}u_1 - f_0}{f_1u_1} - 2\frac{f_{11}f_0}{f_1^2} = \frac{\mu}{u_1} \tag{2.24}$$

$$g \stackrel{\text{def}}{=} \left[f_0 + f_1u_1\frac{\partial}{\partial u} - (f_0u_1 + \lambda)\frac{\partial}{\partial u_1}\right]C(u, u_1) - f_1^{-2}f_{11} \tag{2.25}$$

The first of these equations enables us to find g in terms of the derivatives $f_0 = \partial f/\partial u$, $f_{01} = \partial^2 f/\partial u \partial u_1, \ldots,$ of the function f, the second serves to determine the coefficient $C(u, u_1)$.

For $\lambda = 0$, the equation

$$u_t = -f_1^{-1}(Df)^{-1} + C(u, u_1) \tag{2.26}$$

automatically satisfies the conditions (2.3), (2.4) for an arbitrary choice of the solution $C(u, u_1)$, of Eq. (2.25). The requirement of absence of conservation laws imposes the restriction $\mu \neq 0$ on the right side of Eq. (2.24). For $\mu = 0$ Eq. (2.26) coincides with Eq. (2.21).

For $\lambda \neq 0$, the equation

$$u_t = -f_1^{-1}(Df + \lambda)^{-1} + C(u, u_1) \tag{2.27}$$

with an arbitrary choice of the solution $C(u, u_1)$ of Eq. (2.25) satisfies condition (2.2), but does not satisfy condition (2.3). Condition (2.3) leads to an ordinary differential equation for determining $C(u, u_1)$:

$$f_1^{-1}u_1^{-1}\left[2\lambda \frac{\partial}{\partial u_1} + f_1(\mu - \lambda g)\right]C + \frac{2f_{11}}{f_1^3 u_1} - \frac{g^2}{2} + \frac{2g}{u_1 f_1} + \frac{g_1}{f_1} = \mu_1 \tag{2.28}$$

For arbitrary $\mu_1 \in \mathbb{C}$ the solution of (2.28) is a solution of (2.25). Condition (2.4) is satisfied automatically. The criterion for existence of a nontrivial conservation law selects the particular solution

$$C(u, u_1) = -(2\lambda)^{-1} g u_1 + (\lambda f_1)^{-1} - \frac{\mu}{2\lambda^2} u_1$$

of Eq. (2.25). This function satisfies (2.28) for $\mu_1 + (2\lambda^2)^{-1}\mu^2 = 0$. If this relation between the parameters λ, μ, μ_1 is violated, Eq. (2.27) with the function $C(u, u_1)$ from (2.28) has only trivial conservation laws.

2.2 *Classification of equations* $u_t = F(u, u_1, u_2, u_3)$, $\partial F/\partial u_3 = \mathrm{const}$

For the equation

$$u_t = u_3 + g(u, u_1, u_2) \tag{2.29}$$

the first of conditions (1.10) with label $k = -1$ is satisfied automatically, since $\partial F/\partial u_3 = 1$. The next two conditions ($k = 0, 1$) become explicit (cf. the Corollary to Theorem 1.9), i.e., they are expressed directly in terms of the right side of Eqs. (2.29):

$$\partial_F p_0 \in \operatorname{Im} D, \qquad p_0 = g_2 \overset{\text{def}}{=} \partial g/\partial u_2 \tag{2.30}$$

$$\partial_F p_1 \in \operatorname{Im} D, \qquad p_1 = 3 g_1 g_2^2 \tag{2.31}$$

Here and in the following, $g_i = \partial g/\partial u_i$, $g_0 = \partial g/\partial u$. For $k = 2, 3$, the conditions (1.10) can be written in the form

$$\partial_F p_2 \in \operatorname{Im} D, \qquad p_2 = 27 g_0 - 9 g_1 g_2 + 2 g_2^3 + 9 D^{-1} \partial_F g_2 \tag{2.32}$$

$$\partial_F p_3 \in \operatorname{Im} D, \qquad p_3 = D^{-1} \partial_F (3 g_1 - g_2^2) \tag{2.33}$$

Equation (2.29), unlike second order equations, can have an infinite series of conservation laws. By considering the chain of equations (1.16) and sharpening the arguments used in the derivation of conditions (1.17), (1.18), one can prove the following assertion.

LEMMA 2.4 For Eq. (2.29), the density p of the conservation law of order $n \geqslant 2$ admits the representation

$$p = \varphi(u, \ldots, u_{n-1}) u_n^2 + \psi(u, \ldots, u_{n-1})$$

From the existence of a conservation law of second (third) order, it follows that (cf. (2.30), (2.32)) $p_0 \in \operatorname{Im} D$ ($p_0 \in \operatorname{Im} D$, $p_2 \in \operatorname{Im} D + \mathbb{C}$), respectively.

From Lemma (2.4) it follows that the order of the conservation law (2.30) with density $p_0 = g_2$ cannot be equal to two. Therefore

$$g_2 = p(u, u_1) + Df(u, u_1) \tag{2.34}$$

Using (2.34), the density of the conservation law (2.32) can be written in the form

$$p_2 = \left(f_{111} - 2f_1 f_{11} + \tfrac{4}{9} f_1^3\right)u_2^3 + \varphi(u,u_1)u_2^2 + \psi(u,u_1) + \operatorname{Im} D$$

Returning to Lemma 2.4, we find that the coefficient of u_2^3 is equal to zero. This gives

$$f_{111} - 2f_1 f_{11} + \tfrac{4}{9} f_1^3 \equiv \left(D - \tfrac{2}{3} f_1\right)^2 f_1 = 0$$

$$f(u,u_1) = -\tfrac{3}{2}\ln\left[\alpha(u)u_1^2 + \beta(u)u_1 + \gamma(u)\right]$$

A check of the cases $\alpha \neq 0$; $\alpha = 0$, $\beta \neq 0$; $\alpha = \beta = 0$, leads, by virtue of (2.34) to a sharpening of the form of Eq. (2.28):

$$u_t = u_3 - \frac{3}{2}\,\frac{\left[u_1 + c(u)\right]u_2^2}{\left[u_1 + a(u)\right]^2 + b(u)} + A(u,u_1)u_2 + h(u,u_1); \tag{2.35}$$

$$u_t = u_3 - \frac{3}{4}\,\frac{u_2^2}{u_1 + a(u)} + A(u,u_1)u_2 + h(u,u_1); \tag{2.36}$$

$$u_t = u_3 + A(u,u_1)u_2 + h(u,u_1) \tag{2.37}$$

Further specification of the form of the right side is done individually for each of these three cases.

For equations of the form (2.37), we find from condition (2.30) that the function A depends on u_1 linearly. Therefore, by making the invertible substitution $u \leftrightarrow \varphi(u)$ equations of this type are reduced either to the equation

$$u_t = u_3 + \lambda u_2 + h(u,u_1), \qquad \lambda \in \mathbb{C}, \tag{2.37$'$}$$

or the equation

$$u_t = u_3 + c(u)u_2 + h(u,u_1), \qquad c' \neq 0. \tag{2.37$''$}$$

For Eq. (2.37$'$) the conditions (2.31), (2.32) mean that

$$\partial_F h_1 \in \operatorname{Im} D, \qquad \partial_F h_0 \in \operatorname{Im} D$$

This permits us to sharpen the form of the function $h(u, u_1)$. From condition (2.33) we find that $\lambda = 0$. As a result we arrive at the following list of equations of the form (2.37′):

$$u_t = u_3 + (\alpha u^2 + \beta u + \gamma)u_1 \tag{2.38}$$

$$u_t = u_3 + \alpha u_1^3 + \beta u_1^2 + \gamma u_1 + \delta \tag{2.39}$$

$$u_t = u_3 - \tfrac{1}{8}u_1^3 + (\alpha e^u + \beta e^{-u} + \gamma)u_1 \tag{2.40}$$

Here $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

For Eq. (2.37″) condition (2.30) enables us to sharpen the form of the function $h(u, u_1)$:

$$h = u_1^2(2c'^2 + 2c''c - c'''u_1)(2c')^{-1} + d(u)u_1$$

The form of the functions $c(u), d(u)$ is found from conditions (2.31) and (2.32). As a result we get the following two equations:

$$u_t = u_3 + \tfrac{3}{2}uu_2 + \tfrac{3}{2}u_1^2 + \tfrac{3}{4}u^2u_1 + \alpha u_1$$

$$u_t = u_3 + u^2u_2 + 3uu_1^2 + \tfrac{1}{3}u^4u_1 + \alpha u_1$$

Both of these equations are reduced to the linear equation $u'_t = u'_3 + \alpha u'_1$ by a differential substitution of the form (1.33)—the first by the substitution that linearizes the Burgers equation (cf. the Introduction), the second by the substitution given in Example 1.21.

Equations of the form (2.37) admit an exhaustive classification. To within a substitution (1.33), any of these equations is equivalent either to the linear equation mentioned above, or to the Korteweg–deVries equation $u'_t = u'_3 + u'u'_1 + \gamma u'_1$. Some quite well known equations (2.38)–(2.40) (cf. for example, [2], [16]) belong to the second type. The explicit form of the substitution $u' = \varphi(u, u_1, u_2)$ is also determined as in Example 1.20.

For an equation of the form (2.36) the substitution $u \leftrightarrow \varphi(u)$ allows us to assume that $a(u) = 1$ or 0. Omitting the quite tiresome running through of the cases that arise here, we give the final answer. For

$a(u) = 0$ there are two equations, satisfying conditions (2.30)–(2.33):

$$u_t = u_3 - \frac{3}{4}\frac{u_2^2}{u_1} + \alpha u \tag{2.41}$$

$$u_t = u_3 - \frac{3}{4}\frac{u_2^2}{u_1} + \alpha u_1^{3/2} + \beta u_1^2 + \gamma \tag{2.42}$$

The case $a(u) = 1$ gives four equations. One of them was given in the Introduction as Eq. (0.11). The remaining three have the following forms:

$$u_t = u_3 - \frac{3}{4}\frac{u_2^2}{u_1 + 1} + \alpha(u_1 + 1)^{3/2} + \beta u_1^2 + \gamma, \tag{2.43}$$

$$u_t = u_3 - \frac{3}{4}\frac{u_2^2}{u_1 + 1} + (u_1 + 1)u_2 + \frac{1}{9}(u_1 + 1)^3 + \alpha, \tag{2.44}$$

$$u_t = u_3 - \frac{3}{4}\frac{u_2^2}{u_1 + 1} - 3u_2(u_1 + 1)^{1/2}u^{-1} - 3u_2(u_1 + 1)u^{-1}$$

$$- 6u_1(u_1 + 1)^{3/2}u^{-2} + 3u_1(u_1 + 1)(u_1 + 2)u^{-2} \tag{2.45}$$

At first glance the difference in the numerical coefficients $(-3/2)$ in (2.35) and $(-3/4)$ in (2.36) appears insignificant. However, from formulas (2.34) and (2.33) it follows that

$$p_3 = \left(f_{11} - \tfrac{2}{3}f_1^2\right)u_3^2 + \varphi(u, u_1, u_2) + \operatorname{Im} D$$

For Eq. (2.35), in contrast to Eqs. (2.36), (2.37), the coefficient $f_{11} - 2f_1^2/3$ of u_3^2 is not equal to zero. From this and Lemma 2.4 it follows that in the case of (2.35) conservation laws (2.30), (2.32) are trivial. In fact, in the present case the condition (2.33) signifies the existence of a conservation law of order three, which is also required in the conditions of Lemma 2.4. The triviality of the conservation laws (2.30), (2.32) is used essentially in the classification of equations of the form (2.35).

Setting $a = 0$ in (2.35) (cf. the case of (2.36)), we can, by a substitution $u \leftrightarrow \varphi(u)$ change $b(u)$ to zero or one. The case $a = b = 0$ gives two equations:

$$u_t = u_3 - \tfrac{3}{2}u_2^2 u_1^{-1} + \alpha u_1^{-1} + \beta \tag{2.46}$$

$$u_t = u_3 - \tfrac{3}{2}u_2^2 u_1^{-1} + \alpha u_1^{-1} - \tfrac{3}{2}\mathscr{P}(u)u_1^3 \tag{2.47}$$

where $(\mathscr{P}')^2 = 4\mathscr{P}^3 + c_1\mathscr{P} + c_2$, $c_i \in \mathbb{C}$. The case of $a = 0$, $b = 1$ gives the following two equations:

$$u_t = u_3 - \frac{3}{2}\,\frac{u_1 u_2^2}{u_1^2 + 1} + \alpha\left(u_1^2 + 1\right)^{3/2} + \beta u_1^3 + \gamma \tag{2.48}$$

$$u_t = u_3 - \frac{3}{2}\,\frac{(u_1 + 1)u_2^2}{u_1^2 + 1} - \frac{3}{2}\mathscr{P}(u)\left(u_1^3 + u_1\right) \tag{2.49}$$

Two more equations are obtained for $a = 1$:

$$u_t = u_3 - \tfrac{3}{2}u_2^2(u_1 + 1)^{-1} + \alpha(u_1 + 1)^{-1} + \beta(u_1 + 1)^3 + \gamma \tag{2.50}$$

$$u_t = u_3 - \frac{3}{2}\,\frac{(u_1 + 1)u_2^2}{(u_1 + 1)^2 + \alpha} + \beta\left[(u_1 + 1)^2 + \alpha\right]^{3/2} + \gamma(u_1 + 1)^3 + \delta \tag{2.51}$$

The seventeen equations enumerated exhaust the list of equations (2.29) satisfying the conditions (2.30)–(2.33). For equations of the form (2.35), (2.36), in contrast to equations of the form (2.37), the classification has not been carried to completion. It is not clear, for example, whether Eq. (1.8) is solvable for any choice of constants in (2.41)–(2.51). We give below model equations of the form (2.35), (2.36) and show differential substitutions relating them to the generalized Korteweg–deVries equation and the linear equation, respectively.

The solvability of (1.8) for the model equations follows from Lemma 1.22:

$$u_t = u_3 - \frac{3}{2}\,\frac{(u_1+\lambda)u_2^2}{(u_1+\lambda)^2+\alpha},$$
$$u' = \ln\left\{u_1+\lambda+\sqrt{(u_1+\lambda)^2+\alpha}\right\}, \qquad u'_t = u'_3 - \tfrac{1}{2}(u'_1)^3; \tag{2.52}$$

$$u_t = u_3 - \frac{3}{4}\,\frac{u_2^2}{u_1+\lambda}, \qquad u' = \sqrt{u_1+\lambda}, \qquad u'_t = u'_3 \tag{2.53}$$

Simplified variants of Eqs. (2.52), (2.53) were considered in the Introduction (Eq. (0.9)) and in Example 1.20.

It should be mentioned that Eq. (2.47) arose first in the work of S. P. Novikov and I. M. Krichever in connection with the Kadomtsev–Petriashvili equation (cf. [16]). Equation (2.49) is a different form of the equation

$$u_t = u_3 + 6u_1\frac{(u_2+u-2u^3)^2}{\lambda-4(u_1^2+u^2-u^4)} - 6u^2u_1$$

found by F. Calogero and A. Degasperis (cf. [17]). The case of degeneracy $\mathscr{P} = u^{-2}$ was omitted in [17]. The value $\lambda = 1$ corresponds to trigonometric degeneracy of the Weierstrass function.

§3 Discrete Evolution Equations

In this Section we consider infinite dynamical systems of the form (0.12). Emphasizing the analogy to the case of differential equations (0.1), we make systematic use of the notation of §1: the dynamical variables are denoted by $u_k = D^k u$, $k \in \mathbb{Z}$; $\mathscr{F}$ denotes the algebra of functions depending on a finite set of the dynamical variables;

$\mathscr{F}((D^{-1}))$ is the algebra of formal series of the form

$$L = \sum_{i=-\infty}^{n} a_i D^i, \qquad a_i \in \mathscr{F}$$

etc. We should remember that, unlike the case for differential equations, in this Section D denotes the shift operator, acting in $\mathscr{F}$ according to the formula

$$D^i : f(u_k, u_{k+1}, \ldots, u_l) \mapsto f(u_{k+i}, u_{k+i+1}, \ldots u_{l+i}).$$

3.1 *Conservation laws and conditions for solvability of operator equations*

A discrete evolution equation of the form (0.12) generates a derivation $\partial_F : \mathscr{F} \to \mathscr{F}$, taking u into F and commuting with D. The derivation ∂_F acts on any function $f(u_k, \ldots, u_l)$ according to the formula

$$\partial_F(f) = f_*(F), \qquad f_* \overset{\text{def}}{=} \sum_{i=k}^{l} \frac{\partial f}{\partial u_i} D^i$$

The function $p \in \mathscr{F}$ is called the density for the conservation law for Eq. (0.12) if the condition

$$\partial_F(p) \in \operatorname{Im}(1 - D)$$

is satisfied. The variational derivative $g = \delta p / \delta u$ of the density of the conservation law satisfies (cf. the derivation of formula (1.13)) the equation

$$\left(\partial_F + F_*'\right) g = 0 \tag{3.1}$$

Formulas for the variational derivative and the operation of taking the formal adjoint in the discrete case have the following form:

$$\frac{\delta p}{\delta u} \overset{\text{def}}{=} \sum_{i=-\infty}^{+\infty} \frac{\partial}{\partial u} D^i(p), \qquad \left(\sum a_i D^i\right)' \overset{\text{def}}{=} \sum D^{-i} \circ a_i \tag{3.2}$$

From (3.2) it follows that the variational derivative $g = \delta p/\delta u$ satisfies the condition

$$(g_*)' = g_*$$

Therefore the function g has the form $g = g(u_{-n}, \ldots, u_n)$, $\partial g/\partial u_{-n} \neq 0$, $\partial g/\partial u_n \neq 0$. We shall call the number n the order of the conservation law with density p.

LEMMA 3.1 If the equation $u_t = F(u_{-m'}, u_{-m'+1}, \ldots, u_m)$, $m > -m'$ has a conservation law of order $n \geqslant \max(m, m')$, then $m' = m$.

Proof Applying the operation $*$ to (3.1), we find (cf. (1.14)) that

$$\left(\partial_F + F_*^t\right) g_* - g_*\left(\partial_F - F_*\right) = \sum_{i=-\infty}^{k} a_i D^i \tag{3.3}$$

where $k \leqslant \max(m, m')$. Comparing the leading coefficients in (3.3), we obtain the assertion of the Lemma.

As in §1.3, the relation (3.3) leads to the operator equation

$$\partial_F(S) + F_*^t S + S F_* = 0 \tag{3.4}$$

Since in the discrete case D and D^{-1} are on an equal footing, we should investigate the solvability of (3.4) both in $\mathscr{F}((D^{-1}))$ and in $\mathscr{F}((D))$. However, if $S \in \mathscr{F}((D^{-1}))$ is a solution of Eq. (3.4), then, as is easily seen, $S^t \in \mathscr{F}(D))$ is also a solution. Therefore it is sufficient to consider the set $\hat{B}(F)$ of solutions of (3.3) from $\mathscr{F}((D^{-1}))$. Just as in §1 (cf. Lemma 1.10) one verifies that two different operators S_1, S_2 in $\hat{B}(F)$ generate the operator $L = S_2^{-1} S_1$ in $\hat{A}(F)$, i.e., a solution in $\mathscr{F}((D^{-1}))$ of the equation

$$\left[\partial_F - F_*, L\right] = 0 \tag{3.5}$$

Example 3.2 Consider the equation

$$u_t = u(u_1 - u_{-1}) \tag{3.6}$$

which is the discrete analog of the Korteweg–deVries equation (0.6).

It is not difficult to check that the operators S_1, S_2, where

$$S_1^{-1} = uDu - (uDu)',$$

$$S_2^{-1} = uDuDu + uDu^2 + u^2Du - (uDuDu + uDu^2 + u^2Du)'$$

are solutions of (3.4). Therefore the operator

$$L = S_2^{-1}S_1 = u\left[u_1 D + u_1 + u + (u_1 - u_{-1})D^{-2}(1 - D^{-1})^{-1}\right]u^{-1} \tag{3.7}$$

satisfies relation (3.5).

Just as in the case of differential equations, the first-order operator $L \in \hat{A}(F)$ generates a series of conservation laws

$$\partial_F(\operatorname{res} L^k) \in \operatorname{Im}(1 - D), \qquad k = \mathbb{N} \tag{3.8}$$

where

$$\operatorname{res} \sum a_i D^i \overset{\text{def}}{=} a_0 \tag{3.9}$$

This follows from the relation

$$\operatorname{res}[P, Q] \in \operatorname{Im}(1 - D)$$

valid for any $P, Q \in \mathcal{F}((D^{-1}))$. The proof of this relation is analogous to the proof of Lemma 1.8.

Example 3.2 (continuation) One can show (cf. [7]) that, in the case of (3.7), the conservation laws (3.8) with densities $p_k = \operatorname{res} L^k$, are nontrivial for all $k = 1, 2, 3, \ldots$, and that the density for any conservation law of Eq. (3.6) is a linear combination of the densities p_k, $k = 0, 1, 2, \ldots$, where $p_0 = \ln u$. Moreover, one verifies that for $k \geqslant 1$, the functions $p_k, \delta p_k/\delta u$ are homogeneous polynomials of degree $k, k-1$. For example,

$$p_2 = \operatorname{res} L^2 = u^2 + 3u_1 u + u_1^2 + u_1 u_2 = 2u^2 + 4uu_1 + \operatorname{Im}(1 - D)$$

$$\delta p_2/\delta u = 4(u + u_{-1} + u_1)$$

In general a conservation law for Eq. (3.6) with density p_k has order $k-1$.

We recall that, for the differential analog (0.6) of Eq. (3.6), conservation laws of the form (3.8) with even index are trivial because of Theorem 1.13.

The essential difference of discrete equations manifests itself in the question of minimal positive order of operators in $\hat{A}(F), \hat{B}(F)$. By virtue of Theorem 1.12, in the differential case there exist operators in $\hat{A}(F), \hat{B}(F)$ of first order. The analog of Theorem 1.12 in the discrete case is

THEOREM 3.3 Let us assume that Eq. (0.12) has an infinite series of conservation laws of orders $k_1, k_2, k_3, \ldots,$ satisfying the condition

$$0 < k_{i+1} - k_i \leqslant N.$$

Then there exist operators $L \in \hat{A}(F)$ and $S \in \hat{B}(F)$ whose orders N' and N'', respectively, do not exceed the number N.

Abbreviated proof We can choose $N', N'' \in \mathbb{N}$ and a subsequence of orders $k_{i_1}, k_{i_2}, \ldots,$ such that

$$k_{i_2} - k_{i_1} = k_{i_4} - k_{i_3} = \cdots = N', \qquad k_{i_j} = N'' \pmod{N'}$$

Just as in Theorem 1.12 one verifies that for any l there exist operators $L_l \in \hat{A}_l$, $S_l \in \hat{B}_l$, of orders N' and N'', respectively. The existence of operators $L \in \hat{A}(F)$ and $S \in \hat{B}(F)$ of orders N' and N'' follows from considerations analogous to the proof of Theorem 1.7.

We denote by G the set of all monomials of the form αD^k, where $\alpha \in \mathscr{F}$, $k \in \mathbb{Z}$. It is clear that G is a group under multiplication. Consider the problem of the structure of the centralizer $[g]$ of an arbitrary element $g \in G$. We call the rank of $[g]$ the smallest positive order of elements of $[g]$.

LEMMA 3.4 Suppose that the rank of $[g]$ is r. Then

$$[g] = \{c_i g_r^i, c_i \in \mathbb{C}, i \in \mathbb{Z}\}$$

where g_r is an arbitrary element of $[g]$ with order r.

Proof We denote by Ω the set of orders of elements of $[g]$. Since $[g]$ is a subgroup of the group G, the set Ω is a subgroup in $\mathbb{Z}$ and, consequently, $\Omega = r\mathbb{Z}$, where r is the rank $[g]$. We verify that for $\alpha_1 D^k, \alpha_2 D^k \in [g]$, the relation $\alpha_1 = c\alpha_2$, $c \in \mathbb{C}$ is valid. Suppose that $g = aD^n$; then

$$aD^n \alpha_i D^k = \alpha_i D^k aD^n \Leftrightarrow aD^n(\alpha_i) = \alpha_i D^k(a), \qquad i = 1, 2.$$

Eliminating a, we find

$$D^n(\alpha_1/\alpha_2) = \alpha_1/\alpha_2 \Rightarrow \alpha_1/\alpha_2 = \text{const}$$

If $\alpha D^k \in [g]$, then $k = rk_1$. The operators αD^k and $g_r^{k_1}$ have the same order and belong to $[g]$. Therefore $\alpha D^k = c g_r^{k_1}$.

Corollary to Lemma 3.4 If $[fD^{mq}, aD^{nq}] = 0$, then there exists a monomial αD^q such that

$$fD^{mq} = c_1(\alpha D^q)^m, \qquad aD^{nq} = c_2(\alpha D^q)^n \tag{3.10}$$

In particular, if $[aD^n, bD^n] = 0$, then $a/b = \text{const}$.

Proposition 3.5 Suppose that r is the smallest positive order of the operators L of $\hat{A}(F)$. Then

$$\hat{A}(F) = \left\{ \sum_{i=-\infty}^{n} c_i L_r^i,\ c_i \in \mathbb{C},\ n \in \mathbb{Z} \right\}$$

where L_r is an arbitrary element of $\hat{A}(F)$ of order r.

Proof Suppose that L is an operator of order k in $\hat{A}(F)$ and

$$L = a_k D^k + a_{k-1} D^{k-1} + \cdots, \qquad F_* = F_m D^m + F_{m-1} D^{m-1} + \cdots$$

where $F_i \overset{\text{def}}{=} \partial F/\partial u_i$. From relation (3.5) it follows that $[a_k D^k, F_m D^m] = 0$. Applying Lemma 3.4, we conclude that the leading coefficients of the operators $L_1, L_2 \in \hat{A}(F)$ are proportional, if $\operatorname{ord} L_1 = \operatorname{ord} L_2$.

We denote by Ω the set of orders of operators in $\hat{A}(F)$. Since $\hat{A}(F)$ is a group under multiplication, $\Omega = r\mathbb{Z}$. Subtracting from $L \in \hat{A}(F)$ the operator $cL_r^{k/r}$, we can lower the order of L. Continuing this process we arrive at the decomposition $L = \sum c_i L_r^i$.

From the proof of Proposition 3.5 it is obvious that the leading monomials $a_k D^k$ of the operators in $\hat{A}(F)$ form a subgroup of the commutative group $[F_m D^m] \subset G$ (the commutativity of the centralizer follows from Lemma 3.4). It is clear that the following assertion is valid.

PROPOSITION 3.6 The rank of the centralizer $[F_m D^m]$, $F_m \overset{\text{def}}{=} \partial F/\partial u_m$, is a divisor of rank of $\hat{A}(F)$ (i.e., the smallest positive rank of operators in $\hat{A}(F)$).

Because of Lemma 3.1, the discrete evolution equation having conservation laws of sufficiently high order is expressed in the form

$$u_t = F(u_{-m}, u_{-m+1}, \ldots, u_m), \qquad F_{-m} \cdot F_m \neq 0 \tag{3.11}$$

We shall call the number m of the order of Eq. (3.11). For the equations (3.11) known to the authors, having a nontrivial algebra $\hat{A}(F)$, the rank of $\hat{A}(F)$ is always a divisor of m. An example of a discrete equation for which the rank of $\hat{A}(F)$ is equal to the order of the equation is the following generalization of (3.6):

$$u_t = u(u_m - u_{-m})$$

obtained by the substitution $u_i \to u_{im}$. It is clear that the operator of order m, obtained from (3.7) by the replacement $u_i \to u_{im}$, $D^i \to D^{im}$, belongs to $\hat{A}(F)$. On the other hand, the rank of the centralizer $[g]$, $g = F_m D^m = uD^m$ is m. Using Proposition 3.6 we find that the rank of $\hat{A}(F)$ equals m.

Remark In the case of differential equations, for which the operator $L \in \hat{A}(F)$ is of order n, there existed the operator $L^{1/n}$ also belonging to $\hat{A}(F)$, so that the rank of $\hat{A}(F)$ was always equal to unity.

Let us consider the question of the conditions for solvability of the operator equation (3.5). In the differential case the first of the solvability conditions was formulated (cf. (1.10)) in terms of $\operatorname{res} L^{-1}$. There is no discrete analog of this condition, since in the discrete case, because of the definition (3.9), $\operatorname{res} L^{-1} = 0$. Let us derive the condition analogous to the second of the conditions (1.10).

PROPOSITION 3.7 From the solvability of Eq. (3.5) it follows that the condition

$$\partial_F\left(\ln \frac{\partial F}{\partial u_m}\right) \in \operatorname{Im}(1 - D) \tag{3.12}$$

is satisfied.

Proof From Eq. (3.5) we find that

$$\operatorname{res}\left(L^{-1}\partial_F(L)\right) = \operatorname{res}\left(L^{-1}F_* L - F_*\right)$$

$$= \operatorname{res}\left[L^{-1}F_*, L\right] \in \operatorname{Im}(1 - D)$$

Suppose that $L = \sum_{i=-\infty}^{n} a_i D^i$; then with $F_m = \partial F/\partial u_m$, we have, because of Lemma 3.4:

$$\left[a_n D^n, F_m D^m\right] = 0 \Rightarrow a_n D^n = c_1(\alpha D^q)^{n_1}, \qquad F_m D^m = c_2(\alpha D^q)^{m_1}$$

where q is the greatest common divisor of the orders n and m. Then

$$\operatorname{res}\left(L^{-1}\partial_F(L)\right) = a_n^{-1}\partial_F a_n = \sum_{i=0}^{n_1-1} D^{iq}\left(\alpha^{-1}\partial_F \alpha\right)$$

$$F_m^{-1}\partial_F(F_m) = \sum_{i=0}^{m_1-1} D^{iq}\left(\alpha^{-1}\partial_F \alpha\right)$$

Noting that $\forall f, g \in \mathscr{F}$

$$f + D(g) \in \operatorname{Im}(1 - D) \Leftrightarrow f + g \in \operatorname{Im}(1 - D) \tag{3.13}$$

we get

$$a_n^{-1}\partial_F a_n \in \operatorname{Im}(1 - D) \Rightarrow \alpha^{-1}\partial_F \alpha \in \operatorname{Im}(1 - D)$$

$$\Rightarrow F_m^{-1}\partial_F F_m \in \operatorname{Im}(1 - D)$$

In the discrete case, as in the differential case, from the solvability of (3.5) it follows that the equation under consideration has a series of conservation laws (3.8). However, conditions (3.12) and (3.8) do not exhaust, even in the case when the rank of $\hat{A}(F)$ is unity, the list of solvability conditions. In fact, from Proposition 3.6 and Lemma 3.4 it follows that

$$F_m D^m = (\alpha D^r)^{m/r} \Rightarrow \ln \frac{D^r F_m}{F_m} = \ln \frac{D^m \alpha}{\alpha} \in \operatorname{Im}(1 - D^m)$$

where r does not exceed the rank of $\hat{A}(F)$, the order m of the equation and, generally speaking, $r < m$. Since $\operatorname{Ker} \delta/\delta u = \operatorname{Im}(1 - D) + \mathbb{C}$, the condition

$$\ln \frac{D^r F_m}{F_m} \in \operatorname{Im}(1 - D^m)$$

implies the relation

$$\sum_{k=-\infty}^{+\infty} \frac{\partial}{\partial u} D^{km} \ln \frac{D^r F_m}{F_m} = 0$$

It is not difficult to verify that this condition is by no means fulfilled for all functions $F_m \in \mathcal{F}$. For example, if $F_m = F_m(u)$, $r = 1$, this condition is fulfilled if and only if $m = 1$.

Let us consider the solvability conditions of the operator equation (3.4). As mentioned in the proof of Lemma 3.1, from the solvability of (3.4) it follows that the discrete equation has the form (3.11). Suppose that $S = \alpha D^n + \cdots$. Equating the coefficients of D^{m+n} in (3.4), we get

$$D^{n-m} \ln F_m - \ln(-F_{-m}) = (1 - D^{-m}) \ln \alpha \tag{3.14}$$

Weakening it, we can make condition (3.14) independent of the order n of the operator S. Noting that $(1 - D^{-m}) \ln \alpha \in \operatorname{Im}(1 - D)$ and using (3.13) we get

$$\ln(-F_m/F_{-m}) \in \operatorname{Im}(1 - D) \tag{3.15}$$

The conditions (3.14) and (3.15) are equivalent for $m = 1$.

Let us find still another form of the condition for solvability of Eq. (3.4), analogous to the condition (3.12). Because of (3.4),

$$\partial_F(S)S^{-1} + F'_* + SF_*S^{-1} = 0$$

Then

$$\operatorname{res} F'_* + \operatorname{res} F_* + \operatorname{res}\big(\partial_F(S)S^{-1}\big) \in \operatorname{Im}(1 - D)$$

or

$$2\frac{\partial F}{\partial u} + \partial_F \ln \alpha \in \operatorname{Im}(1 - D) \tag{3.16}$$

where α is the leading coefficient of the operator S.

3.2 Classification of equations $u_t = F(u_{-1}, u, u_1)$

In this Section we consider discrete evolution equations of the form (3.11) with $m = 1$, satisfying conditions (3.12), (3.15), (3.16). Sharpening the arguments of §3.1, we can verify that for $m = 1$ these conditions are satisfied for any equation (3.1) for which there exists a conservation law of order greater than two and an evolution derivation ∂_f such that $[\partial_f, \partial_F] = 0$, ord $f_* \geqslant 2$. The presentation is based on the work of R. I. Yamilov.

For $m = 1$ the class of Eqs. (3.11) satisfying condition (3.15) can be described as follows.

LEMMA 3.8 The function $F(u_{-1}, u, u_1)$ satisfies (3.15) if and only if it can be represented in the form $F = \varphi(u, \psi)$, where the function $\psi(u_{-1}, u, u_1)$ satisfies the symmetry condition

$$\frac{\partial \psi}{\partial u_1} = -D\left(\frac{\partial \psi}{\partial u_{-1}}\right) \tag{3.17}$$

Proof Suppose that condition (3.15) is satisfied. Then there exists a function $\varphi \in \mathscr{F}$ such that

$$\varphi \frac{\partial F}{\partial u_{-1}} + D^{-1}(\varphi)\frac{\partial F}{\partial u_1} = 0 \tag{3.18}$$

The dependence of φ on $u_{-1}, u_{\pm k}$, $k > 1$ contradicts the structure of relation (3.18) and, consequently, $\varphi = \varphi(u, u_1)$. It is not difficult to verify that a particular solution of the partial differential equation (3.18) is the function

$$\psi(u_{-1}, u, u_1) = \int \varphi \, du_1 - \int D^{-1}(\varphi) \, du_{-1} \tag{3.19}$$

satisfying (3.17). The required relation $F = \varphi(u, \psi)$ follows from the formula for the general solution of Eq. (3.18).

Suppose that $F = \varphi(u, \psi)$, where $\psi(u_{-1}, u, u_1)$ satisfies (3.17). It is not hard to show that functions satisfying (3.17) are expressible in the form

$$\psi = \frac{\partial}{\partial u}(1 - D^{-1})\hat{\psi}(u, u_1) \tag{3.20}$$

and that condition (3.18) for $F = \varphi(u, \psi)$ is satisfied with $\varphi = \partial^2 \hat{\psi}/\partial u \, \partial u_1$.

Remark It is easily checked that condition (3.17) can be rewritten in the form $\psi_* = -\psi_*' + 2\,\partial\psi/\partial u$. The connection of condition (3.17) with the condition $f_* = f_*'$, characterizing functions representable in the form of a variational derivative (3.2), becomes obvious if we compare formula (3.20) with the formula:

$$\frac{\delta}{\delta u} f(u, u_1) = \frac{\partial}{\partial u}(1 + D^{-1}) f(u, u_1)$$

In view of Lemma 3.8, the equation satisfying condition (3.15) can be written in the form

$$u_t = \varphi(u, \psi), \qquad \psi = \frac{\partial}{\partial u}(1 - D^{-1})\hat{\psi}(u, u_1) \tag{3.21}$$

We specify the character of the dependence of the function φ on ψ in (3.21) by means of the condition (3.12):

$$\partial_F\left(\ln \frac{\partial F}{\partial u_1}\right) = \partial_F(\ln \varphi') + \partial_F(\ln \psi_1) \in \operatorname{Im}(1 - D) \tag{3.22}$$

where $\varphi' \overset{\text{def}}{=} \partial\varphi/\partial\psi$, $\psi_1 = \partial\psi/\partial u_1$.

PROPOSITION 3.9 Suppose that Eq. (3.21) satisfies condition (3.22). Then

$$\frac{\partial^2}{\partial\psi^2}\ln\varphi'(u,\psi)=\lambda\varphi'(u,\psi),\qquad \lambda\in\mathbb{C} \tag{3.23}$$

and the condition $\lambda\neq 0$ is the criterion for the existence for Eq. (3.21) of a conservation law of second order.

Proof By virtue of (3.22) Eq. (3.21) has a conservation law of order 0, 1 or 2 (the order of the conservation law was determined before Lemma 3.1). In the contrary case the conservation law (3.22) is trivial, i.e., $\partial F/\partial u_1=(1-D^{-1})g(u,u_1)$. The density $f=f(u_{-2},u_{-1},\ldots,u_2)$ of an arbitrary conservation law of order $k\leqslant 2$ is expressed in the form

$$f=p_1(u_{-2},u_{-1},u)+p_2(u_{-1},u,u_1)+p_3(u,u_1,u_2)$$

$$=p_2+Dp_1+D^{-1}p_3+(1-D)p_1+(1-D^{-1})p_3$$

Consequently $f=p(u,u_1,u_{-1})$ modulo $\mathrm{Im}(1-D)$. Change from the variables u,u_1,u_{-1} to variables $u,u_1,\psi(u_{-1},u,u_1)$ and set $p(u,u_1,u_{-1})=\rho(u,u_1,\psi)$. Using the symmetry property (3.17) we find

$$\frac{\partial}{\partial u_3}\frac{\delta}{\delta u}f_t=\psi_1\left[\psi_1\varphi' D,(\psi_1\partial^2\rho/\partial\psi^2+\partial^2\rho/\partial u_1\,\partial\psi)D\right]D^{-3}$$

The left side is equal to zero, since $f_t\in\mathrm{Im}(1-D)$. Equating the commutator on the right to zero, (cf. the Corollary to Lemma 3.4) leads to the relation

$$\partial^2\rho/\partial\psi^2+\psi_1^{-1}\partial^2\rho/\partial u_1\,\partial\psi=\lambda\varphi' \tag{3.24}$$

Equation (3.23) is obtained as the result of substitution in (3.24) of the density for the conservation law (3.22). For $\lambda=0$ the general solution of Eq. (3.24) is expressed by the formula

$$\rho=p_1(u,u_1)+p_2(u_{-1},u)$$

Consequently, the order of the conservation laws in this case cannot be equal to two. Conversely, for $\lambda \neq 0$ Eq. (3.24) has no solutions corresponding to conservation laws of order $k = 0, 1$ and, therefore, the order of the conservation law (3.22) is equal to two.

Let us discuss briefly the relation between the last condition (3.16) and condition (3.22). One can check that for Eq. (3.21) when conditions (3.14), (3.22) are satisfied, condition (3.16) is equivalent to the following:

$$2\frac{\partial F}{\partial u} + \partial_F \ln \varphi \in \operatorname{Im}(1 - D) \tag{3.25}$$

Here F and φ are related by Eq. (3.18). The relation between the conditions (3.22), (3.25) depends essentially on the specific form of the function $\varphi(u, \psi)$ satisfying Eq. (3.23). For example, for $\varphi = \psi^{-1}$, after analysis it becomes clear that condition (3.22) and (3.25) are equivalent. We consider below the simplest case, $\varphi = \psi$. We first note that conditions (3.18), (3.22), (3.25) are invariant under point transformations $u \leftrightarrow \int g(u)\,du$, and that the formulas

$$\frac{\partial}{\partial u_k} \leftrightarrow g(u_k)\frac{\partial}{\partial u_k}, \qquad F \leftrightarrow gF, \qquad \varphi \leftrightarrow gD(g)\varphi, \qquad \psi \leftrightarrow g\psi$$

allow us to rewrite these conditions in invariant terms.

LEMMA 3.10 For the equation

$$u_t = g^2(u)\psi(u_{-1}, u, u_1) \tag{3.26}$$

a criterion for equivalence of conditions (3.22) and (3.25) is the relation

$$g(u)\frac{\partial}{\partial u}(g(u)\psi) \in \operatorname{Im}(1 - D) \tag{3.27}$$

Proof For $g = 1$, i.e., $F = \psi$, in condition (3.18), it follows that $\varphi = \partial\psi/\partial u_1$. Comparing (3.22), (3.25) we get (3.27). The general case reduces to that considered above by the point transformation $u \leftrightarrow \int g(u)\,du$.

From the point of view of classification, a distinguished case is that of equations having conservation laws of zeroth order (cf. §2.1).

PROPOSITION 3.11 The equation $u_t = F(u_{-1}, u, u_1)$, having a conservation law of zero order and satisfying conditions (3.18), (3.22), (3.25), is equivalent, up to the group of point transformations, to one of the following two equations:

$$u_t = (\alpha u^2 + \beta u + \gamma)(u_1 - u_{-1}) \tag{3.28}$$

$$u_t = (\alpha u^4 + \beta u^2 + \gamma)\left(\frac{1}{u_1 + u} - \frac{1}{u + u_{-1}}\right) \tag{3.29}$$

Here $\alpha, \beta, \gamma \in \mathbb{C}$.

Proof Suppose that $p(u)$ is the density of a conservation law; then $p'(u)F = (1 - D^{-1})g(u, u_1)$. Because of (3.18) we have

$$\varphi^{-1}\frac{\partial g}{\partial u_1} = D^{-1}\left(\varphi^{-1}\frac{\partial g}{\partial u}\right)$$

From this, setting $\varphi^{-1}\partial g/\partial u_1 = \int h(u)\,du$, we find

$$g(u, u_1) = f[h(u) + h(u_1)].$$

After the substitution $u \leftrightarrow h(u)$ we obtain

$$u_t = a(u)[f(u + u_1) - f(u_{-1} + u)] \tag{3.30}$$

Formula (3.30) determines the general form of equations that have a conservation law of zero order and satisfy the condition (3.18). The conservation law (3.22) enables us to specify the form of the function f on the right side of Eq. (3.30). For density $p(u, u_1)$ of an arbitrary conservation law of order no higher than one, following the derivation of (3.24), we find

$$\partial^2 p/\partial u\,\partial u_1 = 2\mu f'(u + u_1), \qquad \mu \in \mathbb{C}$$

Substitution in this of the density of the conservation law (3.22) gives

$$f' = \mu f^2 + \epsilon f + \delta \tag{3.31}$$

From (3.31) it follows that Eq. (3.30) is brought by a point transformation to the form

$$u_t = a(u)(u_1 - u_{-1}) \tag{3.28'}$$

for $\mu = 0$, while for $\mu \neq 0$ it takes the form

$$u_t = a(u)\left[\frac{1}{u_1 + u} - \frac{1}{u + u_{-1}}\right] \tag{3.29'}$$

For example, as is easily checked, the equation

$$u_t = a(u)\left[\operatorname{cth}(u_1 + u) - \operatorname{cth}(u + u_{-1})\right]$$

corresponding the case of $\mu \neq 0$, $\epsilon^2 \neq 4\mu^2\delta$, reduces to the form (3.29′) under the substitution $u \leftrightarrow \operatorname{cth} u$.

For Eq. (3.28′), the condition (3.27) gives

$$\begin{aligned} a'(u)u_1 - a'(u)u_{-1} & \\ &= a'(u)u_1 - a'(u_1)u - (1 - D)\{a'(u)u_{-1}\} \in \operatorname{Im}(1 - D) \end{aligned}$$

From this we find

$$a'(u)u_1 - a'(u_1)u = (1 - D)p(u) \Rightarrow a''(u) = a''(u_1)$$

and, consequently, $a'' = \text{const}$, $a(u) = \alpha u^2 + \beta u + \gamma$.

Analogously, for Eq. (3.29′) it follows from (3.27) that

$$\begin{aligned} 12\left[a(u_1) - a(u)\right] - 6\left[a'(u_1) - a'(u)\right](u_{-1} + u) & \\ + \left[a''(u_1) - a''(u)\right](u_1 + u)^2 = 0 & \end{aligned}$$

Then we obtain $d^4a/du^4 = \text{const}$, and thus $a(u) = \alpha u^4 + \beta u^2 + \gamma$. Fulfillment of conditions (3.22), (3.25) for Eqs. (3.28), (3.29) can be verified directly.

Returning to Proposition 3.9, we give the general form of Eqs. (3.21), satisfying condition (3.23). We here take into account the nonuniqueness of the choice of the function ψ in (3.21), related to the gauge transformation $\psi \to \psi + q(u)$, which is allowed by condition

(3.17). For $\lambda = 0$, Eq. (3.23) gives, as a complement to (3.30),

$$u_t = g(u)\left[\exp\{ g(u)\psi\} + p(u)\right] \tag{3.32}$$

For $\lambda \neq 0$ we get two more equations:

$$u_t = \psi^{-1} + \mu g(u), \qquad \mu \in \mathbb{C} \tag{3.33}$$

$$u_t = g(u)\left[\operatorname{th}\{ g(u)\psi\} + p(u)\right] \tag{3.34}$$

A complete list of Eqs. (3.21), satisfying conditions (3.22), (3.25) has at present been obtained only for $\lambda = 0$. In addition to Eqs. (3.28), (3.29) this list contains the two equations:

$$u_t = \left(\alpha u^4 + \beta u^3 + \gamma u^2 + \delta u + \epsilon\right)\left(\frac{1}{u - u_1} + \frac{1}{u_{-1} - u}\right) \tag{3.35}$$

$$u_t = g(u_1 - u)g(u - u_1) + \epsilon, \qquad g' = \alpha g + \beta + \gamma g^{-1} \tag{3.36}$$

Equation (3.35) is, roughly speaking, a generalization of Eq. (3.29), and for $\beta = \delta = 0$ reduces to Eq. (3.29) under the invertible transformation

$$u_k \leftrightarrow (-1)^k u_{-k}, \qquad k = 0, \pm 1, \ldots$$

Equation (3.36) is unique up to the group of point transformations in the class of Eqs. (3.32), and reduces to Eq. (3.28) under the substitution $u' = g(u_1 - u)$. This substitution is a characteristic example of discrete analogs of the differential substitutions that were considered in §1.4.

Examples of equations of the form (3.33) and (3.34) are Eqs. (0.13) and (0.14), respectively.

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