Russian Academy of Sciences
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Encyclopedia of integrable systems

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The Encyclopedia contains

basics of the theory of nonlinear integrable systems;

tests of integrability and lists of integrable systems based on their intrinsic properties;

actual information on particular equations.

The Encyclopedia is a free irregularly renewed edition. We invite specialists to submit articles on the subject, as well as remarks, corrections and suggestions.

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Italic marks the terms without separate entries.

Equation labels:

e	evolutionary
h	hyperbolic
d	dispersionless
q	quantum
D	differential (subscripts denote derivatives)
Δ	difference (subscripts denote shifts)

Colors:

D	integrable
D	linearizable
D	not integrable

List of authors

1 Ablowitz–Ladik lattice

$$u_{n,t} = u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}), \quad -v_{n,t} = v_{n+1} - 2v_n + v_{n-1} + u_n v_n (v_{n+1} + v_{n-1})$$

Alias: Discrete NLS

➢ Introduced in [1] as the discretization of NLS equation.

➢ Reduction $t \to it$, $v = \bar{u}$: $iu_t = u_1 - 2u + u_{-1} - |u|^2(u_1 + u_{-1})$.

➢ The lattice represents the linear combination of three commuting flows which are members of one integrable hierarchy:

$$u_{n,x_0} = u_n, \quad v_{n,x_0} = -v_n, \qquad u_{n,x_{\pm1}} = u_{n\pm1}(1 + u_n v_n), \quad v_{n,x_{\pm1}} = -v_{n\mp1}(1 + u_n v_n).$$

➢ Hamiltonian structure:

$$\{u_n, v_n\} = 1 + u_n v_n, \quad H_{\pm1} = \sum u_{n\pm1} v_n, \quad H_0 = \sum \log(1 + u_n v_n).$$

➢ Zero curvature representation $L_{n,x_k} = U^{(k)}_{n+1} L_n - L_n U^{(k)}_n$:

$$L_n = \begin{pmatrix} \lambda^{-1} & -v_n \\ u_n & \lambda \end{pmatrix}, \quad U^{(1)} = \begin{pmatrix} 0 & -\lambda v_{n-1} \\ \lambda u_n & u_n v_{n-1} + \lambda^2 \end{pmatrix},$$

$$U^{(-1)} = \begin{pmatrix} -v_n u_{n-1} - \lambda^{-2} & v_n/\lambda \\ -u_{n-1}/\lambda & 0 \end{pmatrix}, \quad -2U^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

➢ For each n, the variables u_n, v_n satisfy the Pohlmeyer–Lund–Regge system

$$u_{x_+x_-} = \frac{v u_{x_+} u_{x_-}}{uv + 1} + u(uv + 1), \quad v_{x_+x_-} = \frac{u v_{x_+} v_{x_-}}{uv + 1} + v(uv + 1).$$

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2 Ablowitz–Ladik lattice multifield

$$u_{n,t} = u_{n+1} - 2u_n + u_{n-1} + u_{n-1}v_nu_n + u_nv_nu_{n+1},\qquad u_n \in \mathrm{Mat}(M,N),\quad v_n \in \mathrm{Mat}(N,M) \tag{1}$$
$$-v_{n,t} = v_{n+1} - 2v_n + v_{n-1} + v_{n-1}u_nv_n + v_nu_nv_{n+1},$$

$$u_{n,t} = u_{n+1} - 2u_n + u_{n-1} + \langle u_n, v_n\rangle(u_{n+1} + u_{n-1}),\qquad u_n, v_n \in \mathbb{R}^N \tag{2}$$
$$-v_{n,t} = v_{n+1} - 2v_n + v_{n-1} + \langle u_n, v_n\rangle(v_{n+1} + v_{n-1}),$$

Like in the scalar case, the lattice (1) represents the linear combination of the commuting flows

$$u_{n,x} = u_nv_nu_{n+1} + u_{n+1},\qquad -v_{n,x} = v_{n-1}u_nv_n + v_{n-1} \tag{3}$$
$$u_{n,y} = u_{n-1}v_nu_n + u_{n-1},\qquad -v_{n,y} = v_nu_nv_{n+1} + v_{n+1} \tag{4}$$
$$u_{n,z} = u_n,\qquad -v_{n,z} = v_n,$$

however, the symmetry $x \leftrightarrow y$, $n \to -n$ disappears. The flows (3), (4) correspond to the matrix generalization of Pohlmeyer–Lund–Regge system of the form

$$u_{xy} = u_yv(uv+1)^{-1}u_x + uvu + u,\quad v_{xy} = v_x(uv+1)^{-1}uv_y + vuv + v.$$

In particular, the vector case $M = 1$ gives rise to the equations

$$u_{n,x} = (\langle u_n, v_n\rangle + 1)u_{n+1},\qquad -v_{n,x} = (\langle u_n, v_n\rangle + 1)v_{n-1},$$
$$u_{n,y} = \langle u_{n-1}, v_n\rangle u_n + u_{n-1},\qquad -v_{n,y} = \langle u_n, v_{n+1}\rangle v_n + v_{n+1},$$
$$u_{xy} = \frac{\langle u_y, v\rangle u_x}{\langle u, v\rangle + 1} + \langle u, v\rangle u + u,\qquad v_{xy} = \frac{\langle u, v_y\rangle v_x}{\langle u, v\rangle + 1} + \langle u, v\rangle v + v.$$

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3 Adler–Kostant–Symes scheme

Author: V.V. Sokolov, 09.02.2009

1. Factorization method

Adler–Kostant–Symes scheme [1, 2] (also known as ***factorization method***) allows to integrate an ODE system of the following special form:

$$U_t = [U_+, U], \quad U(0) = U_0. \tag{1}$$

Here $U(t)$ is a function with the values in a *Lie algebra* $\mathfrak{G}$ decomposed into the direct sum of vector subspaces $\mathfrak{G}_+$ and $\mathfrak{G}_-$:

$$\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-, \tag{2}$$

each subspace being a subalgebra in $\mathfrak{G}$. The notation U_+ means the projection of U onto $\mathfrak{G}_+$. We will assume, for the sake of simplicity that $\mathfrak{G}$ is embedded into the matrix algebra.

The solution of the Cauchy problem (1) is given by the formula

$$U(t) = A(t)U_0A^{-1}(t) \tag{3}$$

with the matrix $A(t)$ is defined as the solution of the factorization problem

$$A^{-1}B = \exp(-U_0t), \quad A \in G_+, \quad B \in G_-, \tag{4}$$

where G_+ and G_- are Lie groups of the algebras $\mathfrak{G}_+$ and $\mathfrak{G}_-$. If $\mathfrak{G}_-$ is ideal then the factorisation problem is solved explicitly:

$$A = \exp((U_0)_+t), \quad B = A\exp(-U_0t).$$

In the case when the groups G_+ and G_- are algebraic, the conditions $A \in G_+$ and $A\exp(-U_0t) \in G_-$ form a system of algebraic equations which define $A(t)$ uniquely (at t in the nearby of zero). We will demonstrate below (see (7)) that in the case when the corresponding Lie groups are not algebraic the factorization problem can be reduced to a linear differential equation with variable coefficients.

The most known decomposition (2) of the matrix algebra $\mathfrak{G} = \mathrm{Mat}_N$ is the ***Gauss decomposition*** with the space of upper-triangular matrices as $\mathfrak{G}_+$ and the space of lower-triangular matrices with zero diagonal as $\mathfrak{G}_-$. The corresponding factorization problem (4) is easily solved by means of linear algebra. A less trivial is ***Iwasawa decomposition*** with the spaces of upper-triangular matrices as $\mathfrak{G}_+$ and of skew-symmetric ones as $\mathfrak{G}_-$.

A more general factorization problem

$$A^{-1}B = Z(t), \quad Z(0) = E, \quad A \in G_+, \quad B \in G_- \tag{5}$$

is closely related to equations of the form

$$U_t = [U_+, U] + F(U), \quad U(0) = U_0 \tag{6}$$

where $F : \mathfrak{G} \to \mathfrak{G}$ is a mapping invariant with respect to the group G_+ (a simplest mapping of such kind is $F(U) = \lambda U$, $\lambda = \mathrm{const}$). Namely, let Z satisfies the linear equation

$$Z_t = q(t)Z, \quad Z(0) = E,$$

then the formula

$$U(t) = Aq(t)A^{-1}$$

solves the equation (6) if

$$q_t = F(q), \quad q(0) = U_0.$$

Thus, if one is able to solve this Cauchy problem then the factorization method allows to solve the problem (6) as well.

The factorization problem (5) can be reduced [3] to a linear equation with variable coefficients. Let us define the linear mapping $L(t) : \mathfrak{G} \to \mathfrak{G}$ as follows

$$L(t)(v) = \big(Z^{-1}(t)vZ(t)\big)_+.$$

Since $L(0)$ is the identity map, hence $L(t)$ is invertible for small t. Let A be the solution of Cauchy problem

$$A_t = -AL^{-1}(t)\big((Z^{-1}Z_t)_+\big), \quad A(0) = E, \tag{7}$$

then the pair A, $B = AZ(t)$ is the unique solution of the factorization problem (5).

2. Reductions and nonassociative algebras

It is clear from (3) that if the initial data U_0 belong to a vector space M which is $\mathfrak{G}_+$-module then $U(t) \in M$ for all t. We call such a specialization of the (1) as ***M-reduction***. The possibility of reductions greatly extends the frames of the factorization method (see e.g. [4]).

There are deep relations between M-reductions and several classes of nonassociative algebras [5, 4]. Let $R : M \to \mathfrak{G}_+$ denote the projector onto $\mathfrak{G}_+$ parallel to $\mathfrak{G}_-$. It terms of the operator R, the M-reduction is written as

$$m_t = [R(m), m], \quad m \in M. \tag{8}$$

Let us consider the algebraic operation on M defined by formula

$$m * n = [R(m), n]. \tag{9}$$

The system (8) takes, in terms of this multiplication, the form

$$m_t = m * m. \tag{10}$$

Let us show that if the multiplication $*$ is left-symmetric then the system (10) is integrable by factorization method. Let $\mathfrak{A}$ be a left-symmetric algebra. Let $\mathfrak{G} = \mathfrak{A} \oplus \mathfrak{A}$. Since the operation $[X, Y] = X * Y - Y * X$ defines a Lie algebra for any left-symmetric algebra $\mathfrak{A}$, hence the vector space $\mathfrak{G}$ becomes the Lie algebra with respect to the bracket

$$\big[(g_1, a_1), (g_2, a_2)\big] = \big([g_1, g_2], g_1 * a_2 - g_2 * a_1\big).$$

It is clear from this formula that $\mathfrak{G}_+ = \{(q, 0)\}$ and $\mathfrak{G}_- = \{(q, -q)\}$ are subalgebras in $\mathfrak{G}$. The equation (1) for $U = (p, q)$ corresponding to the decomposition $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-$ is of the form

$$p_t = q * p - p * q, \quad q_t = p * q + q * q.$$

In order to obtain the $\mathfrak{A}$-top (10) as a M-reduction of this system it is sufficient to set $p = 0$, that is, to choose $M = \{(0, q)\}$.

It should be noted that the operation (9) is left-symmetric if and only if the operator $R : M \to \mathfrak{G}_+$ satisfies the relation (cf [6])

$$R([R(a), b] + [a, R(b)]) - [R(a), R(b)] \in \mathrm{Ann}(M)$$

where a, b are any elements of M and $\mathrm{Ann}(M)$ denotes the set of $\mathfrak{G}_+$ elements with zero image of M.

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4 Algebraic structures

The set G equipped with the multiplication $G \times G \to G$ is called ***group*** if the following identities are fulfilled:

$$\begin{array}{lll} \textit{associativity} & \forall\, a,b,c & a(bc) = a(bc), \\ \textit{unit element} & \exists\, e:\ \forall\, a & ea = ae = a, \\ \textit{inverse element} & \forall\, a\ \exists\, a^{-1}: & a(bc) = a(bc). \end{array} \tag{1}$$

An ***algebra*** is a vector space A over a field F, equipped with a multiplication $A \times A \to A$ which satisfies the identities

$$(\alpha a + \beta b)c = \alpha ac + \beta bc, \quad c(\alpha a + \beta b) = \alpha ca + \beta cb, \quad \forall\, a,b,c \in A, \quad \forall\, \alpha, \beta \in F.$$

The important classes of algebras are characterized by some additional identities, for example:

$$\begin{array}{ll} \textit{commutative algebra} & ab = ba \\ \textit{anticommutative algebra} & ab = -ba \\ \textit{associative algebra} & a(bc) = (ab)c \\ \textit{Lie algebra} & ab = -ba, \quad a(bc) + b(ca) + c(ab) = 0 \\ \textit{Jordan algebra} & ab = ba, \quad (ab)a^2 = a(ba^2) \\ \textit{left-symmetric algebra} & a(bc) - (ab)c = b(ac) - (ba)c \end{array}$$

An important example of an algebraic structure with ternary multiplication is given by *Jordan pairs*.

A linear mapping $F : A \to A$ is called a ***differentiation*** of an algebra A if it satisfies the ***Leibniz rule***

$$F(ab) = F(a)b + aF(b).$$

The set of all differentiations of the algebra is denoted $\mathrm{Der}(A)$. It is a Lie algebra itself with respect to the commutator $[F,G](a) = F(G(a)) - G(F(a))$. Indeed,

$$[F,G](ab) = F(G(a)b + aG(b)) - G(F(a)b + aF(b))$$

$$= F(G(a))b + G(a)F(b) + F(a)G(b) + aF(G(b)) - G(F(a))b - F(a)G(b) - G(a)F(b) - aG(F(b))$$
$$= [F, G](a)b + a[F, G](b),$$

and it is easy to check that the Jacobi identity is fulfilled.

5 Bäcklund transformation

Bäcklund transformation between equations $F[u]=0$, $G[\hat{u}]=0$ is a set of relations $A[u,\hat{u}]=0$, $B[u,\hat{u}]=0$ which satisfy the property that elimination of $\hat{u}$ yields the given equation for u and vice versa. The most important is the case when the equations coincide (or differ in the values of parameters). In this case the term ***Bäcklund autotransformation*** is used. Iterations of auto-BT generate the differential-difference equations, or lattices.

➢ The simplest example is given by the Cauchy–Riemann equations $u_x=v_y$, $u_y=-v_x$; here both u and v satisfy the Laplace equation $u_{xx}+u_{yy}=0$.

➢ The first nontrivial nonlinear example was introduced by L. Bianchi and A.V. Bäcklund in the 1880's. Geometrically, it describes a transformation of pseudospherical surfaces. Analytically, it can be brought to the pair of relations

$$\hat{u}_x+u_x=a\sin(\hat{u}-u),\quad \hat{u}_y-u_y=\frac{1}{2a}\sin(\hat{u}+u) \tag{1}$$

and one can easily check that both u and $\hat{u}$ satisfy, in virtue of these relations, the sine-Gordon equation

$$u_{xy}=\sin 2u.$$

Let $u=u_n$ and $\hat{u}=u_{n+1}$, then the x-part of this auto-BT gives rise to the lattice

$$u_{n+1,x}+u_{n,x}=a_n\sin(u_{n+1}-u_n)$$

which is an example of the so-called dressing chains.

➢ In all known examples, the construction of BT is somehow related with the auxiliary linear problems associated with the equation under consideration. The most important BT are Darboux transformations which make use of a particular solution of the linear problems. On the nonlinear level such transformation is usually given by Riccati-type equations, like in (1). Another type of BT is given by explicit transformations like the invertible substitution

$$\hat{u}=u_x/u+v,\quad \hat{v}=u$$

which acts on the solutions of the Levi system

$$u_t=u_{xx}+(u^2+2uv)_x,\quad v_t=-v_{xx}+(v^2+2uv)_x.$$

This substitution generates (again, let $u = u_n$ and $\hat{u} = u_{n+1}$) the Volterra lattice

$$u_{n,x} = u_n(u_{n+1} - u_{n-1}).$$

➢ The term ***Bäcklund transformation*** is also widely used in the theory of Painlevé-type ODE. In this context it denotes a rational differential substitution between equations with different parameter sets. For example, the second Painlevé equation

$$u'' = 2u^3 + zu + a$$

admits the pair of BT

$$\hat{u} = u \pm \frac{2a \pm 1}{2u' \pm 2u^2 \pm z}, \quad \hat{a} = \pm 1 - a$$

which allows to generate solutions for all values of the parameter $a + 2n$, $-a + 2n + 1$, $n \in \mathbb{Z}$.

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6 Bakirov system

$$u_t = 5u_4 + v^2, \quad v_t = v_4$$

The only higher symmetry of this system is

$$u_t = 11u_6 + 5vv_2 + 4v_1^2, \quad v_t = v_6.$$

Bakirov checked that there are no other symmetries up to order 53. The rigorous proof was obtained in [2].

➢ See also: Fokas conjecture.

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7 Belousov–Zhabotinsky model

$$\dot{u} = av(1-u) + au(1-bu), \quad \dot{v} = -\frac{1}{a}v(1+u) + cw, \quad \dot{w} = d(u-w)$$

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8 Belov–Chaltikian lattices

$$u_{n,x}^{(j)} = u_n^{(j)}(u_{n+j}^{(1)} - u_{n-1}^{(1)}) + u_n^{(j+1)} - u_{n-1}^{(j+1)}, \quad j = 1, \ldots, M, \quad u_n^{(M+1)} = 0.$$

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9 Benjamin–Bona–Mahoney–Peregrine equation

$$u_t + u_x - u_{xxt} + uu_x = 0$$

Alias: regularized long wave equation

➢ As the famous KdV equation, this one describes one-dimensional long waves of small amplitude [1].

➢ Some (nonintegrable) generalizations in any dimension were studied in [5].

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10 Benjamin–Ono equation

$$u_t + H(u_{xx}) - 6uu_x = 0, \quad H(f) = \frac{1}{\pi} v.p. \int_{-\infty}^{\infty} \frac{f(y)}{y-x} dy$$

Operator H is called the Hilbert transform.

The equation describes one-dimensional waves in deep water.

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11 Benney chain

$$u_{n,t} = u_{n+1,x} + n u_{n-1} u_{0,x}, \quad n = 0, 1, 2, \dots$$

➢ Dispersionless Lax pair $D_t(L) = A_p L_x - A_x L_p$:

$$A = \frac{p^2}{2} + u_0, \quad L = p + u_0 p^{-1} + u_1 p^{-2} + u_2 p^{-3} + \dots$$

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12 Benney equation

$$u_t + uu_x - u_y \int_0^y u_x dy + h_x = 0, \quad h_t + D_x\Big(\int_0^h u dy\Big) = 0$$

References

[1] V.E. Zakharov. On the Benney's equations. *Physica D* ***3*** (1981) 193–200.

13 Bogoyavlensky–Narita lattices

$$u_{n,x_k} = u_n \sum_{s=1}^{k} (u_{n+s} - u_{n-s}) \tag{1}$$

➢ Introduced in [1, 2].

➢ The flow corresponding to x_k does not commute with the rest flows of the family, rather it serves as the simplest member of an integrable hierarchy on its own. The next order flows and associated systems are (n is omitted):

$$\begin{cases} u_{x_1} = u(u_1 - u_{-1}) \\ u_{t_1} = u(u_1(u_2 + u_1 + u) - u_{-1}(u + u_{-1} + u_{-2})) \end{cases} \to \begin{cases} u_{1,t_1} = D_{x_1}(u_{1,x_1} + u_1(u_1 + 2u_0)) \\ u_{0,t_1} = D_{x_1}(-u_{0,x_1} + (2u_1 + u_0)u_0) \end{cases}$$

(this is Volterra lattice and Levi system);

$$\begin{cases} u_{x_2} = u(u_2 + u_1 - u_{-1} - u_{-2}) \\ u_{t_2} = u(u_2(u_4 + \dots + u) + u_1(u_3 + \dots + u) \\ \qquad - u_{-1}(u + \dots + u_{-3}) - u_{-2}(u + \dots + u_{-4})) \end{cases} \to \begin{cases} u_{3,t_2} = D_{x_2}(u_{3,x_2} + u_3(u_3 + 2u_2 + 2u_1)) \\ u_{2,t_2} = D_{x_2}(u_{2,x_2} + u_2(u_2 + 2u_1 + 2u_0)) \\ u_{1,t_2} = D_{x_2}(-u_{1,x_2} + (2u_3 + 2u_2 + u_1)u_1) \\ u_{0,t_2} = D_{x_2}(-u_{0,x_2} + (2u_2 + 2u_1 + u_0)u_0); \end{cases}$$

$$\begin{cases} u_{x_3} = u(u_3 + u_2 + u_1 - u_{-1} - u_{-2} - u_{-3}) \\ u_{t_3} = u(u_3(u_6 + \dots + u) + u_2(u_5 + \dots + u) \\ \qquad + u_1(u_4 + \dots + u) - u_{-1}(u + \dots + u_{-4}) \\ \qquad - u_{-2}(u + \dots + u_{-5}) - u_{-3}(u + \dots + u_{-6})) \end{cases} \to \begin{cases} u_{5,t_3} = D_{x_3}(u_{5,x_3} + u_5(u_5 + 2u_4 + 2u_3 + 2u_2)) \\ u_{4,t_3} = D_{x_3}(u_{4,x_3} + u_4(u_4 + 2u_3 + 2u_2 + 2u_1)) \\ u_{3,t_3} = D_{x_3}(u_{3,x_3} + u_3(u_3 + 2u_2 + 2u_1 + 2u_0)) \\ u_{2,t_3} = D_{x_3}(-u_{2,x_3} + (2u_5 + 2u_4 + 2u_3 + u_2)u_2) \\ u_{1,t_3} = D_{x_3}(-u_{1,x_3} + (2u_4 + 2u_3 + 2u_2 + u_1)u_1) \\ u_{0,t_3} = D_{x_3}(-u_{0,x_3} + (2u_3 + 2u_2 + 2u_1 + u_0)u_0) \end{cases}$$

and so on.

➢ Bogoyavlensky lattices admit a lot of modifications. Some of the difference substitutions for this type of lattices can be described as follows. Let a lattice be given

$$u_{n,x} = u_n f(u_n), \quad f = a^{(k)}T^k + \cdots + a^{(-k)}T^{-k}, \tag{2}$$

where f is a Laurent polynomial on the shift operator $T : u_n \to u_{n+1}$. This polynomial can be factored in many ways into the product of two Laurent polynomials and any such factorization generates the substitution from (1) to the lattice (2)

$$v_{n,x} = v_n h(e^{g(\log v_n)}) \qquad \xrightarrow{u_n = e^{g(\log v_n)}} \qquad u_{n,x} = u_n f(u_n), \quad f = gh.$$

It is easy to see that the lattice for the variables v_n is polynomial if and only if all coefficients of the polynomial g are nonnegative integers, moreover, the total degree of its r.h.s. is equal to the sum of the coefficients of g plus 1.

Notice that the polynomial f for the Bogoyavlensky lattice (1) is

$$f = T^k + \cdots + T - T^{-1} - \cdots - T^{-k} = \frac{(T^k - 1)(T^{k+1} - 1)}{(T - 1)T^k}.$$

Example 1. Consider the lattice

$$u_{n,x} = u_n(u_{n+2} + u_{n+1} - u_{n-1} - u_{n-2}),$$

corresponding to the polynomial $f = T^2 + T - T^{-1} - T^{-2}$. Several possible choices of g and the corresponding substitutions are:

$$\begin{array}{lll} g = T + 1 & u_n = v_{n+1}v_n & v_{n,x} = v_n(v_{n+2}v_{n+1} - v_{n-1}v_{n-2}); \\ g = T^2 + T + 1 & u_n = v_{n+2}v_{n+1}v_n & v_{n,x} = v_n^2(v_{n+2}v_{n+1} - v_{n-1}v_{n-2}); \\ g = T^3 - 1 & u_n = v_{n+3}/v_n & v_{n,x} = v_n(v_{n+2}/v_{n-1} + v_{n+1}/v_{n-2}). \end{array}$$

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14 Boltzman equation

$$u_t = uu_2 + u_1^2$$

The equation is not integrable. The first necessary condition (23.2), (23.3) is not fulfilled:

$$\rho_{-1} = u^{-1/2}, \quad D_t(\rho_{-1}) \notin \operatorname{Im} D_x.$$

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15 Born–Infeld equation

$$(1 - u_t^2)u_{xx} + 2u_x u_t u_{xt} - (1 + u_x^2)u_{tt} = 0$$

- Lagrangian: $L = (1 - u_t^2 + u_x^2)^{1/2}$.
- See also: the minimal surfaces equation

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16 Boussinesq equation

$$u_{tt} = -(u_{xx} + u^2)_{xx}$$

➢ Lax pair [3, 4]:

$$\psi_{xxx} + \frac{3}{2}u\psi_x + \frac{3}{4}(u_x + v)\psi = \lambda\psi, \quad \psi_t = \psi_{xx} + u\psi \qquad \Rightarrow \qquad u_t = v_x, \quad -3v_t = u_{xxx} + 6uu_x.$$

➢ Boussinesq equation can be equivalently written as the NLS type system

$$u_t = u_{xx} + (u + v)^2, \quad -v_t = v_{xx} + (u + v)^2.$$

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17 Boussinesq system, twodimensional

$$u_t = u_{xx} + 2v_x, \quad -v_t = v_{xx} - 2uu_x + 2u_y$$

Elimination of v yields the equation

$$u_{tt} = (u_{xxx} + 4uu_x - 4u_y)_x$$

which coincides with Kadomtsev–Petviashvili equation up to the scaling and exchange $y \leftrightarrow t$.

18 Box-ball system

$$x_n^t \in \{0,1\}, \qquad \sum_{n=-\infty}^{\infty} x_n^t < \infty, \qquad x_n^{t+1} = \begin{cases} 1 & \text{if} \quad x_n^t = 0 \quad \text{and} \quad \sum\limits_{k=-\infty}^{n-1} (x_k^t - x_k^{t+1}) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Alternatively, this cellular automaton can be described as follows. Let 0 represents an empty box and 1 a box with a ball. The number of the balls is finite. Enumerate them from left to right and successively move to the nearest right empty box.

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19 Boyer–Finley equation

$$u_{xx} + u_{yy} = e^{u_t} u_{tt}$$

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20 Burgers equation

$$u_t = u_{xx} + 2uu_x$$

➢ This equation is probably the simplest nonlinear model with applications in hydrodynamics, gas dynamics and acoustic.

➢ The potential Burgers hierarchy ($u = v_1$) can be defined explicitly by formula

$$v_{t_n} = (D_x + v_1)^n(1) = Y_n(v_1, \dots, v_n), \quad v_k = D_x^k(v)$$

where Y_n are ***Bell polynomials***. The following formula for their generating function is easily proven by differentiation with respect to z and x:

$$1 + \sum_{n=1}^{\infty} Y_n \frac{z^n}{n!} = \exp\Big(\sum_{n=1}^{\infty} v_n \frac{z^n}{n!}\Big) = e^{v(x+z)-v(x)}.$$

➢ The ***Cole–Hopf transformation*** [2, 3]

$$v = \log\phi \quad \Rightarrow \quad u = \phi_x/\phi$$

linearizes the whole hierarchy:

$$1 + \sum_{n=1}^{\infty} Y_n \frac{z^n}{n!} = \frac{\phi(x+z)}{\phi(x)} = 1 + \sum_{n=1}^{\infty} \frac{\phi_n}{\phi} \frac{z^n}{n!}.$$

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21 Burgers–Huxley equation

$$u_t = u_{xx} + uu_x + u(u-1)(u-a)$$

Not integrable.

➢ See also: Fischer, Kolmogorov–Petrovsky–Piskunov equations.

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22 Burgers–Korteweg–de Vries equation

$$u_t = au_{xxx} + bu_{xx} + uu_x$$

This equation serves as the simplest model for one-dimensional nonlinear waves in the media with dispersion and dissipation. It has some applications in plasma physics for the description of collisionless shock waves. In contrast to both Burgers equation ($a = 0$) and KdV equation ($b = 0$) this one is *not* integrable.

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23 Burgers-type equations, the classification

Author: V.E. Adler, 29.03.2007

Burgers type equations are integrable evolutionary equations of the second order

$$u_t = F(u_2, u_1, u, x), \quad u_n = D_x^n(u). \tag{1}$$

Here we present their exhaustive classification obtained by Svinolupov. The proof of the following theorem can be converted into a test of integrability applicable to a given equation of the form (1). Moreover, if the equation turns out to be integrable then the change of variables can be found constructively which relates it to one of the equations from the list.

Theorem 1 (Svinolupov [1]). *If equation (1) possesses a higher symmetry of order ≥ 3 then it possesses an infinite algebra of higher symmetries and is contact equivalent to one of the following equations, linearizable via differential substitutions (f denotes an arbitrary function):*

$$u_t = u_2 + f(x)u, \tag{B_1}$$

$$u_t = D_x(u_1 + u^2 + f(x)), \tag{B_2}$$

$$u_t = D_x\left(\frac{u_1}{u^2} - 2x\right), \tag{B_3}$$

$$u_t = D_x\left(\frac{u_1}{u^2} + \varepsilon_1 xu + \varepsilon_2 u\right). \tag{B_4}$$

1. The necessary integrability conditions

Accordingly to the general theory (see formal symmetry test), the necessary integrability conditions are of the form of the conservation laws

$$D_x(\sigma_k) = D_t(\rho_k), \quad k = -1, 0, 1, \dots \tag{2}$$

where the densities ρ_k are algorithmically expressed through the right hand side of the equation and the previously defined σ_i. For the equations (1) we will need only first three conditions.

Lemma 2. *If the equation (1) possesses a higher symmetry of order ≥ 3 then the equations (2) at $k = -1, 0, 1$ are fulfilled with*

$$\rho_{-1} = F_{u_2}^{-1/2}, \quad \rho_0 = F_{u_1} F_{u_2}^{-1} - \sigma_{-1} F_{u_2}^{-1/2},$$
$$\rho_1 = \frac{1}{8}(4F_u + 2\sigma_0 + \sigma_{-1}^2) F_{u_2}^{-1/2} - \frac{1}{32}(2F_{u_1} - D_x(F_{u_2}))^2 F_{u_2}^{-3/2}. \tag{3}$$

2. The analysis of the first necessary condition

In the case of equations (1), the analysis of the integrability conditions is simplified due to the following lemma.

Lemma 3. *The order of a conservation law of the equation (1) is equal to 0 or 2.*

Proof. The conservation law satisfies the equation

$$(D_t + F_*^{\intercal})\Big(\frac{\delta\rho}{\delta u}\Big) = 0$$

where

$$\frac{\delta\rho}{\delta u} = \rho_u - D_x(\rho_{u_1}) + D_x^2(\rho_{u_2}) - \cdots + (-D_x)^m(\rho_{u_m}) = a(x, u, \ldots, u_m)u_{2m} + \ldots.$$

Collecting the terms with u_{2m+2} yields

$$aD_x^{2m}(F) + D_x^2(aF_{u_2}u_{2m}) + \cdots = 0 \tag{4}$$

and if $2m > 2$ then $2aF_{u_2}u_{2m+2} = 0$, hence $a = 0$. ■

Moreover, the order of the conservation law determines the dependence of F on u_2. Indeed, if the equation possesses a conservation law of order 2, then, accordingly to (4),

$$2F_{u_2} + u_2 F_{u_2 u_2} = 0 \quad \Leftrightarrow \quad F = (fu_2 + g)^{-1} + h \tag{5}$$

where f, g, h depend on x, u, u_1. If the order of a conservation law is 0 then equation is quasilinear: let $\rho = \rho(x, u)$, $\rho_u \neq 0$, then

$$D_t(\rho) = \rho_u F \in \operatorname{Im} D_x \quad \Rightarrow \quad F = f(x, u, u_1)u_2 + g(x, u, u_1). \tag{6}$$

The equation with another dependence of the right hand side on u_2 cannot possess the nontrivial conservation law at all.

Now let us consider the first integrability condition $D_t(F_{u_2}^{-1/2}) \in \operatorname{Im} D_x$. The quantity $F_{u_2}^{-1/2}$ is called the ***separant*** of the equation. Accordingly to the Lemma 3 it must be linear in u_2. Three cases are possible:

$$\begin{aligned}
&1) \qquad F_{u_2}^{-1/2} = D_x(\alpha(x, u, u_1)),\\
&2) \qquad F_{u_2}^{-1/2} = D_x(\alpha(x, u, u_1)) + \beta(x, u), \quad \beta_u \neq 0,\\
&3) \qquad F_{u_2}^{-1/2} = D_x(\alpha(x, u, u_1)) + \beta(x, u, u_1), \quad \beta_{u_1 u_1} \neq 0.
\end{aligned}$$

In the case 1) the conservation law is trivial, and in the cases 2), 3) its order is equal, respectively, to 0 and 2. The functions α and β are not independent. Since β is the density of the conservation law, hence

$$D_t(\beta(x, u, u_1)) \sim (\beta_u - D_x(\beta_{u_1}))F \in \operatorname{Im} D_x \quad \Rightarrow \quad \partial_{u_2}^2((\beta_u - D_x(\beta_{u_1}))F) = 0.$$

The last equation is equivalent to

$$\beta_{u_1 u_1}(\alpha_x + \alpha_u u_1 + \beta) = \alpha_{u_1}(\beta_{u_1 x} - \beta_u + \beta_{u_1 u} u_1). \tag{7}$$

In particular, the function α does not depend on u_1 in the case 2), while $\alpha_{u_1} \neq 0$ in the case 3). This is also clear from the formulae (6), (5).

Lemma 4. *The equation (1) satisfies the condition* $D_t(F_{u_2}^{-1/2}) \in \operatorname{Im} D_x$ *if and only if it is contact equivalent to one of the quasi-linear equations*

$$u_t = u_2 + f(x, u, u_1), \tag{8}$$

$$u_t = D_x\Big(\frac{u_1}{u^2} + f(x, u)\Big). \tag{9}$$

Proof. Accordingly to the formula (36.2), the contact transformation

$$\tilde{t} = t, \quad \tilde{x} = \varphi(x, u, u_1), \quad \tilde{u} = \psi(x, u, u_1), \tag{10}$$

$$(\varphi_x + \varphi_u u_1)\psi_{u_1} = (\psi_x + \psi_u u_1)\varphi_{u_1} \tag{11}$$

acts on the separant as follows:

$$F_{u_2}^{-1/2} = D_x(\varphi)\tilde{F}_{\tilde{u}_2}^{-1/2}.$$

In the case 1), the separant can be set to 1. To do this, it is sufficient to define $\varphi = \alpha$ and to find ψ from the equation (11). After this we come, omitting tilde, to an equation of the form (8).

In the cases 2), 3) the separant can be taken as $\tilde{u}$. Since the integrability conditions are invariant with respect to the contact transformations, hence the right hand side of the transformed equation is the total derivative on x and the formula (9) takes place.

The desired transform is the point one in the case 2): it is sufficient to choose the functions $\varphi(x, u)$, $\psi(x, u)$ such that

$$D_x(\alpha(x, u)) + \beta(x, u) = \psi(x, u)D_x(\varphi(x, u)) \quad \Leftrightarrow \quad \alpha_x + \beta = \psi\varphi_x, \quad \alpha_u = \psi\varphi_u.$$

In other words φ should be any non-constant solution of the equation $(\alpha_x + \beta)\varphi_u = \alpha_u\varphi_x$, and ψ is defined as $\psi = (\alpha_x + \beta)/\varphi_x = \alpha_u/\varphi_u$. The Jacobian of the transform is equal to $\beta_u \neq 0$.

Analogously in the case 3), the desired contact transform is defined by equations (11) and

$$\alpha_x + \alpha_u u_1 + \beta = \psi(\varphi_x + \varphi_u u_1), \quad \alpha_{u_1} = \psi\varphi_{u_1}. \tag{12}$$

At the first glance, this system for φ and ψ is overdetermined. However, it turns out to be consistent in virtue of the constraint (7). To demonstrate this, differentiate the first equation (12) with respect to u_1 and, using the second equation and (11), bring the equations (12) to the form

$$\psi\varphi_x = \alpha_x + \beta - u_1\beta_{u_1}, \quad \psi\varphi_u = \alpha_u + \beta_{u_1}, \quad \psi\varphi_{u_1} = \alpha_{u_1} \quad \Rightarrow$$

$$\varphi_x = \frac{\alpha_x + \beta - u_1\beta_{u_1}}{\alpha_{u_1}}\varphi_{u_1}, \quad \varphi_u = \frac{\alpha_u + \beta_{u_1}}{\alpha_{u_1}}\varphi_{u_1}, \quad \psi = \frac{\alpha_{u_1}}{\varphi_{u_1}}.$$

The equation (11) is fulfilled in virtue of this system, and the cross-differentiation yields exactly the equation (7). The corresponding contact transformation is nondegenerate: $w = \psi_u - \psi_{u_1}\varphi_u/\varphi_{u_1} = -\beta_{u_1u_1}/\varphi_{u_1} \neq 0$. ■

3. The conclusion of the proof

The Lemma 4 resolves effectively the first integrability condition and reduces the general problem to the quasilinear one. The further analysis is relatively easy. We perform it separately for the cases (8) and (9).

Proof of Theorem 1. 1) Consider equations of the form (8) first. The canonical densities take the form

$$\rho_0 = f_{u_1}, \quad \rho_1 \sim \frac{1}{2} f_u + \frac{1}{4}\sigma_0.$$

Since the quasilinear equation can possess only zero order conservation laws, hence the density ρ_0 must be linear in u_1, so that the equation is of the form

$$u_t = u_2 + a(x,u)u_1^2 + b(x,u)u_1 + c(x,u).$$

This subclass is invariant with respect to the changes $\tilde{x} = x$, $\tilde{u} = \psi(x,u)$, and the coefficient a is transformed accordingly to the formula $\psi_u^2 \tilde{a}(x,\psi) = \psi_u a(x,u) - \psi_{uu}$. Therefore, the equation can be brought to the form

$$u_t = u_2 + b(x,u)u_1 + c(x,u).$$

Consider the condition $D_t(b) \in \operatorname{Im} D_x$:

$$D_t(b) = b_u(u_2 + bu_1 + c) = D_x\Big(b_u u_1 + \frac{1}{2}b^2\Big) - u_1 D_x(b_u) - bb_x + b_u c \in \operatorname{Im} D_x.$$

It is easy to see that it is equivalent to

$$b = p(x)u + q(x), \quad up'' - (pu+q)(p'u+q') + pc \in \operatorname{Im} D_x.$$

Notice, that we may still use the changes

$$\tilde{u} = \mu(x)u + \nu(x) \quad \Rightarrow \quad p = \mu\tilde{p}, \quad q = 2\mu'/\mu + \nu\tilde{p} + \tilde{q}.$$

Therefore, the function p can be made constant, and q can be set to zero. After this, if $p \neq 0$, then $c = c(x)$ and we obtain the equation (B_2). If $p = 0$ then the function c is determined by the third integrability condition. In this case $f_u = c_u$ must be the density of the conservation law, that is

$$D_t(c_u) = c_{uu}(u_2 + c) \in \operatorname{Im} D \quad \Leftrightarrow \quad D_x^2(c_{uu}) + c_{uuu}(u_2 + c) + c_{uu}c_u = 0 \quad \Leftrightarrow \quad c_{uu} = 0.$$

The change $\tilde{u} = u + \nu(x)$ brings to the equation (B_1).

2) Now, consider the equations of the form (9). In this case the second integrability condition takes the form

$$D_t(u^2 f_u - uf) \in \operatorname{Im} D_x. \tag{13}$$

We have, for the density of the form $\rho = \rho(x, u)$,

$$D_t(\rho) = \rho_u D_x(u^{-2}u_1 + f) \sim -D_x(\rho_u)(u^{-2}u_1 + f) \in \operatorname{Im} D_x \quad \Rightarrow \quad \rho_{uu} = 0,$$

and therefore

$$f = p(x)u + q(x) + r(x)/u.$$

The transform

$$\tilde{x} = \varphi(x), \quad \tilde{u} = u/\varphi'(x)$$

does not change the form of the equation and maps the coefficient f into $\tilde{f} = f + \varphi''/(\varphi' u)$. Use of this transform allows to set $r = 0$, without loss of generality. To do this, it is sufficient to define φ as a nonconstant solution of the equation $\varphi'' = -r\varphi'$. Then the condition (13) is reduced to the following one:

$$-D_t(qu) \sim q'(u^{-2}u_1 + pu) \sim q''u^{-1} + q'pu \in \operatorname{Im} D_x \quad \Rightarrow \quad q'' = q'p = 0.$$

If $q' \neq 0$ then $p = 0$ and the scaling $\tilde{x} = kx$, $\tilde{u} = u/k$ brings to the equation (B_3).

If $q' = 0$ then p should be specified by use of the third integrability condition. In this subcase it takes the form $4\rho_1 = 18u^{-3}u_1^2 - 9u^{-2}u_2 - 3p'u - pu_1 \sim -2p'u$. Therefore,

$$-D_t(p'u) \sim p''(u^{-2}u_1 + pu) \sim p'''u^{-1} + p''pu \in \operatorname{Im} D_x \quad \Rightarrow \quad p'' = 0,$$

and this corresponds to equation (B_4). ■

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24 Calogero equation

$$u_{xt} = uu_{xx} + \Phi(u_x)$$

Liouville type equation.

➢ Particular case: Hunter–Saxton equation [3, 4]

$$u_{xt} = uu_{xx} + \varepsilon u_x^2$$

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25 Calogero–Degasperis equation, elliptic

$$u_t = u_3 - \frac{3u_1 u_2^2}{2(u_1^2+1)} - \frac{3}{2}\wp(u)u_1(u_1^2+1) - 2au_1, \quad \dot{\wp}^2 = 4\wp^3 + g_1\wp + g_2$$

26 Calogero–Degasperis equation, exponential

$$u_t = u_3 - \frac{1}{2}u_1^3 - \frac{3}{2}(e^{2u} + ae^{-2u} + b)u_1$$

27 Calogero–Moser model

$$\ddot{q}_k = -\sum_{j\neq k} f'(q_k - q_j), \quad j,k = 1,\dots,n, \qquad f(x) = \begin{cases} gx^{-2} & \text{rational case} \\ \sinh^{-2} x & \text{hyperbolic case} \\ \wp(x) & \text{elliptic case} \end{cases}$$

➢ Lax pair $\dot{L} = [L, A]$ for the rational case [2]:

$$L_{ij} = p_i\delta_{ij} + \frac{(-g)^{1/2}}{q_i - q_j}(1-\delta_{ij}), \quad p_i = \dot{q}_i, \qquad (-g)^{-1/2}A_{ij} = \delta_{ij}\sum_{k\neq i,j}\frac{1}{(q_i-q_k)^2} - (1-\delta_{ij})\frac{1}{(q_i-q_j)^2}.$$

See also: Ruijsenaars–Schneider model

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28 Camassa–Holm equation

$$u_t - u_{xxt} + 2ku_x = uu_{xxx} + 2u_x u_{xx} - 3uu_x$$

➢ Zero curvature representation:

$$\psi_{xx} = \Big(\lambda(u - u_{xx} + k) + \frac{1}{4}\Big)\psi, \quad \psi_t = \frac{u_x}{2}\psi + \Big(\frac{1}{2\lambda} - u\Big)\psi_x$$

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29 Cellular automata

In the wide sense, a ***cellular automaton*** is a dynamical system with time and spatial independent variables taking integer values and dependent variables taking values in some finite set. In the narrow sense, it is required that the dynamics is described locally. This means that the rules of transition $t \to t+1$ must be determined by the values of dependent variables in some neighborhood of any node of the spatial lattice.

➢ Example: box-ball system.

30 Chen–Lee–Liu system

$$u_t = u_{xx} + 2uvu_x, \quad v_t = -v_{xx} + 2uvv_x$$

Alias: DNLS-II

➢ Bäcklund transformation:

$$u_{n,x} = (u_n v_{n+1} + \beta_n)(u_{n+1} - u_n), \quad v_{n,x} = (u_{n-1}v_n + \beta_{n-1})(v_n - v_{n-1})$$

➢ Nonlinear superposition principle:

$$\tilde{u}_n = u_n + (\beta_{n+1} - \beta_n)\frac{u_{n-1} - u_n}{\beta_n + u_{n-1}v_{n+1}}, \quad \tilde{v}_n = v_n - (\beta_{n+1} - \beta_n)\frac{v_{n+1} - v_n}{\beta_{n-1} + u_{n-1}v_{n+1}}$$

➢ Zero curvature representation:

$$U = \begin{pmatrix} r & \lambda u \\ \lambda v & -r \end{pmatrix}, \quad V = 2rU + \begin{pmatrix} \frac{1}{2}(u_x v - uv_x) & \lambda u_x \\ -\lambda v_x & \frac{1}{2}(uv_x - u_x v) \end{pmatrix}, \quad 2r = uv - \lambda^2$$

$$L_n = (u_n v_{n+1} + \beta_n)^{-\frac{1}{2}} \begin{pmatrix} u_n v_{n+1} + \beta_n - \lambda^2 & \lambda u_n \\ \lambda v_{n+1} & \beta_n \end{pmatrix}$$

➢ A multifield generalization [2, 3]: let u, v belong to an associative algebra, then the system

$$u_t = u_{xx} + 2u_x vu, \quad v_t = -u_{xx} + 2vuv_x$$

possesses the third order symmetry

$$u_{t_3} = u_{xxx} + 3u_{xx}vu + 3u_x vu_x + 3u_x vuvu, \quad v_{t_3} = v_{xxx} - 3vuv_{xx} - 3v_x uv_x + 3vuvuv_x.$$

In the case $u \in \mathrm{Mat}_{M,N}$, $v \in \mathrm{Mat}_{N,M}$ the $M \times M$ matrices

$$U = -2u_x v, \quad W = 2u_x v_x - 2u_{xx}v - 4u_x vuv$$

satisfy the matrix KP equation

$$4U_{t_3} = U_{xxx} - 3(U_xU + UU_x - W_t + [W,U]), \quad W_x = U_t$$

while the $N \times N$ matrices

$$F = vu, \quad P = vu_x - v_xu + F^2$$

satisfy the matrix mKP equation

$$4F_{t_3} = F_{xxx} + 3([F_{xx}, F] - 2FF_xF + P_t + [P, F^2] + F_xP + PF_x), \quad F_t = P_x + [P, F].$$

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31 Chiral fields equation

$$u_x = [u, Jv], \quad v_y = [v, Ju], \quad u, v \in \mathbb{R}^3, \quad |u| = |v| = 1, \quad J = \operatorname{diag}(a, b, c).$$

The linear in λ Lax pair found in [1] (up to the change to light-cone variables; $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$):

$$U = \begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & u_3 & -u_2 \\ -u_2 & -u_3 & 0 & u_1 \\ -u_3 & u_2 & -u_1 & 0 \end{pmatrix} \tilde{J}, \quad V = \begin{pmatrix} 0 & v_1 & v_2 & -v_3 \\ -v_1 & 0 & v_3 & v_2 \\ -v_2 & -v_3 & 0 & -v_1 \\ v_3 & -v_2 & v_1 & 0 \end{pmatrix} \tilde{J},$$

$$\tilde{J} = \lambda I - \frac{1}{2} \operatorname{diag}(c - a - b, b - a - c, a - b - c, a + b + c).$$

Bäcklund transformation and discretization were found in [2].

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32 Chiral fields equation, principal

$$2u_{xy} = u_x u^{-1} u_y + u_y u^{-1} u_x, \quad u \in G$$

where G is a Lie group. The equation can be equivalently rewritten as

$$(u^{-1}u_x)_y + (u^{-1}u_y)_x = 0.$$

➢ Zero curvature representation $U_t - V_x = [V, U]$:

$$U = \frac{u_x u^{-1}}{1-\lambda}, \quad V = \frac{u_y u^{-1}}{1+\lambda}$$

33 Classical symmetry

A ***classical symmetry*** is a local one-parametric group of point or contact transformations which preserve the equation under scrutiny. This notion is wide applicable to all sorts of partial and ordinary differential/difference equations.

The theory of classical symmetries was developed by Lie. The modern treatment of the classical and the generalized evolutionary symmetries can be found in the references below.

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34 Collapse

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The scenarios of collapse in the Cauchy problem for Nonlinear Schrödinger equation

$$iu_t = \Delta u + |u|^\rho u \quad \Rightarrow \quad \frac{d}{dt}\int |u|^2 dx = 0, \quad x \in \mathbb{R}^n$$

were studied in [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13]. The use of Sobolev embedding theorems allows to prove at $n = 2, 3$ the existence of the local solutions of this problem in the following cases:

$$n = 2,\ 1 \le \rho \quad \text{or} \quad n = 3,\ 1 \le \rho < 4 \quad \Rightarrow \quad u \in C([0, t_0)) \cap W_2^2 \cap \{u | ru \in L_2\}.$$

Let us use the energy conservation law

$$E(t) = \int |u_x|^2 dx - \frac{2}{\rho + 2}\int |u|^{\rho+2} dx = E_0 = \text{const}, \tag{1}$$

$$\phi(t) \le 4E_0 t^2 + 4\mu t + \phi(0), \quad \phi := \int r^2 |u|^2 dx.$$

The second term in (1) can be estimated as follows, for $u \in W_2^{1;0}(\mathbb{R}^n)$, $n \ge 2$ and $0 \le \alpha < 1$:

$$||u||_q \le \beta ||u_x||^\alpha ||u||^{1-\alpha}, \quad \frac{1}{q} = \frac{1}{2} - \frac{\alpha}{n}, \quad \text{and} \quad ||u||_6 \le \beta ||u_x||^{1/3} ||u||_4^{2/3}, \quad n = 2 \tag{2}$$

and this allows to prove the global solvability of the Cauchy problem at $\rho < \rho_0$, $\rho_0 = \frac{4}{n}$. Indeed, due to (2)

$$\int |u|^q dx = ||u||_q^q \le \beta ||u_x||^{q\alpha} ||u||^{q - q\alpha} \quad \text{and} \quad q\alpha = 1 \quad \Rightarrow \quad \rho = q - 2 = \frac{4}{n}.$$

At $\rho = \rho_0$ the problem on the collapse admits an explicit solution [14]

$$u(t, x) = (t_0 - t)^{-n/4} v(\xi) \exp\Big(i\frac{\alpha r^2 + \beta t}{t_0 - t}\Big), \quad \xi = \frac{r}{t_0 - t}, \quad v(\xi) > 0,$$

$$v_{\xi\xi}+\frac{n-1}{\xi}v_\xi+\lambda v=0,\quad \inf\{||w||^p:||\nabla w||^2-\frac{2}{\sigma}||w||_\sigma^\sigma\leq 0\},\quad \sigma=\rho+2,\quad \rho=\frac{4}{n}.$$

This scenario is not unique (see multi-particle solutions in [15, 16, 17]).

At $\rho\neq\rho_0$ the problem on the collapse does not admit selfsimilar solutions which vanish at infinity. Similarity Ansatz yields

$$|u|^2=(t_0-t)^{-n/2}\xi^{1-n}A_\xi(\xi),\quad \xi:=\frac{r}{\sqrt{t_0-t}},$$

$$A_{\xi\xi\xi}=\frac{A_{\xi\xi}^2}{2A_\xi}+A_\xi\left(\frac{n^2-4n+3}{2\xi^2}+\frac{\xi^2}{4}+2\left(\frac{A_\xi}{\xi^{n-1}}\right)^{\rho/2}+c\right)+\varepsilon\frac{\xi A}{2}+\varepsilon^2\frac{A^2}{8A_\xi},\quad \varepsilon=\frac{4}{\rho}-n.$$

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35 Conservation law

Conservation law is an equality of the form $\operatorname{div} F = 0$ which turns into identity on any solution of a given PDE. Conservation law is called trivial

1) either if F itself vanishes on the solutions

2) or if $\operatorname{div} F$ vanishes identically (independently on the equation).

Clearly, all conservation laws form a linear space and it is only the factor-space what makes sense, modulo trivial conservation laws. The order of the conservation law is defined as the minimal order with respect to derivatives in the class of equivalence.

➢ Example: consider equation of the form $u_{xt} = f'(u)$. It admits the conservation laws

$$D_t(u_{xt}^2 - f'(u)^2 + u_x) = D_x(u_t), \quad D_t(u_x^2) = D_x(2f(u)).$$

The first equality is a combination of two types of trivial conservation laws; the second one is nontrivial conservation law of first order.

➢ In the case of evolutionary PDE the time-component is often written separately, so that conservation laws take the form $D_t(\rho) = \operatorname{div}_x(\sigma)$, where function ρ is called density and vector σ is called flux of the conservation law. Integration over some spatial domain yields, under suitable boundary condition, the integral constant of motion $\int_\Omega \rho\, dx = \text{const}$.

The notion of the order can be formalized by use of variational derivative. In the simplest case of scalar evolutionary equations with one spatial variable, the order of a conservation law with the density $\rho = \rho(x, u, \dots, u_m)$ is equal to one half of the order of the expression

$$\frac{\delta \rho}{\delta u} = \rho_u - D(\rho_{u_1}) + D^2(\rho_{u_2}) - \dots + (-D)^m(\rho_{u_m}).$$

This does not depend on the addition of type 2) trivial conservation laws, and type 1) is excluded by requirement that ρ does not depend on t-derivatives.

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36 Contact transformations

In the case of one dependent variable ($m = 1$) the point transformations can be generalized as follows. Let $p = (p_1, \dots, p_n)$, $p_i = \partial u/\partial x_i$. The ***contact transformation*** is a nondegenerate transformation of the form

$$X_i = X_i(x,u,p), \quad U = U(x,u,p), \quad P_i = P_i(x,u,p) \tag{1}$$

where functions X_i, U, P_i are related in such a way that the total differential is preserved:

$$dU - \sum P_i dX_i = c(du - \sum p_i dx_i), \quad c = c(x,u,p) \neq 0.$$

More explicitly, this condition is equivalent to the following set of equations:

$$\{X_i, X_j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{P_i, X_j\} = \delta_{ij} c, \quad \{X_i, U\} = 0, \quad \{P_i, U\} = cP_i,$$

where

$$\{f,g\} = \sum_{i=1}^{n} (f_{p_i}(g_{x_i} + p_i g_u) - g_{p_i}(f_{x_i} + p_i f_u)).$$

Note the relation

$$[X_f, X_g] = X_{\{f,g\}}$$

which is valid for the ***contact vector fields***

$$X_f = \sum_i f_{p_i}\partial_{x_i} - \sum_i (f_{x_i} + p_i f_u)\partial_{p_i} + \Big(\sum_i p_i f_{p_i} - f\Big)\partial_u.$$

➢ ***Example. Point and contact transformations of the evolutionary equations.*** As an illustrative example, consider in more details the case of evolutionary $1+1$-dimensional equations

$$u_t = F(t,x,u,u_1,\dots,u_n), \quad u_k = D_x^k(u).$$

It is easy to show that the subgroup of the contact transformations preserving the evolutionary form is given by the formulae

$$T = T(t), \quad X = X(t,x,u,u_1), \quad U = U(t,x,u,u_1)$$

where the functions T, X, U satisfy the conditions

$$T' \neq 0, \quad D_x(X) \neq 0, \quad w = U_u - \frac{U_{u_1}}{X_{u_1}} X_u \neq 0, \quad X_{u_1}(U_x + U_u u_1) = U_{u_1}(X_x + X_u u_1).$$

The prolongation of this transformation on the x-derivatives is given by the formula

$$U_1 = \frac{U_x + U_u u_1}{X_x + X_u u_1} = \frac{U_{u_1}}{X_{u_1}}, \quad U_k = U_k(t, x, u, \ldots, u_k) = D_X^k(U), \quad D_X = \frac{1}{D_x(X)} D_x$$

and the equation $U_T = F(T, X, U, \ldots, U_n)$ transforms into

$$u_t = f(t, x, u, \ldots, u_n) = w^{-1}(T'F - U_t + U_1 X_t).$$

The following formula is valid for the transformations of this type at $k > 1$:

$$U_{k,u_k} = \frac{w}{(D_x(X))^k}. \tag{2}$$

In particular, at $n \geq 2$

$$f_{u_n} = T'(D_x(X))^{-n} F_{U_n}$$

The formula (2) it is valid also at $k = 1$ for the subgroup of the point transformations

$$T = T(t), \quad X = X(t, x, u), \quad U = U(t, x, u),$$

and it is valid also at $k = 0$ ($w = U_u$) for the transformations of the form

$$T = T(t), \quad X = X(t, x), \quad U = U(t, x, u).$$

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37 Darboux transformation

Let us consider the ***Sturm–Liouville spectral problem***

$$\psi_{xx} = (u(x) - \lambda)\psi. \tag{1}$$

Statement 1 (***Darboux transformation*** [1]). *Equation (1) is form invariant under the transformation*

$$\hat{\psi} = \psi_x - f\psi, \quad \hat{u} = u - 2f_x, \quad f := \psi_x^{(\alpha)}/\psi^{(\alpha)} \tag{2}$$

where $\psi^{(\alpha)}$ *is a particular solution of (1) at* $\lambda = \alpha$.

The function f satisfies the Riccati equations

$$f_x + f^2 = u - \alpha, \quad -f_x + f^2 = \hat{u} - \alpha.$$

The iteration of Darboux transformation brings to the sequence of operators $L_n = -D_x^2 + u_n$, $A_n = -D_x + f_n$ related by equations

$$L_n = A_n^+ A_n + \alpha_n \quad \to \quad L_{n+1} = A_n A_n^+ + \beta_n = A_{n+1}^+ A_{n+1} + \alpha_{n+1}.$$

This sequence is governed by the dressing chain

$$f_{n,x} + f_{n+1,x} = f_n^2 - f_{n+1}^2 + \alpha_n - \alpha_{n+1}.$$

Any solution of this differential-difference equation generates a family of the operators L_n with ψ-functions calculated for all $\lambda = \beta_k$ explicitly:

$$\psi_{k,k} = \exp(\int f_k dx), \quad \psi_{n,k} = A_n^+ \psi_{n+1,k}, \quad n < k. \tag{3}$$

This feature explains the role of Darboux transformation in quantum mechanics, see factorization method.

Darboux transformation admits straightforward generalizations for linear problems of any order and in any dimension. We mention here few most typical examples. In particular, the above transform can be easily obtained by separation of variables from the following one.

Statement 2. *The 2-dimensional Schrödinger equation*

$$\sigma\psi_y = \psi_{xx} - u(x,y)\psi \tag{4}$$

is form invariant under the transformation

$$\hat\psi = \psi_x - f\psi, \quad \hat u = u - 2f_x, \quad f := \phi_x/\phi \tag{5}$$

where ϕ is any particular solution of (4).

Proof. Denote $g = \phi_y/\phi$ then $g_x = f_y$, $\sigma g = f_x + f^2 - u$ and

$$L = \sigma D_y - D_x^2 + u = \sigma(D_y - g) - (D_x + f)(D_x - f), \quad [D_y - g, D_x - f].$$

Therefore $\hat\psi$ satisfies the equation $\hat L\hat\psi = 0$, where $\hat L = \sigma(D_y - g) - (D_x - f)(D_x + f) = L - 2f_x$. ■

Iterations of the transform (5) are governed by the 2D dressing chain

$$f_{n,x} + f_{n+1,x} = f_n^2 - f_{n+1}^2 - \sigma(g_n - g_{n+1}), \quad g_{n,x} = f_{n,y}.$$

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38 Darboux system

$$u^{ij}_{x_k} = u^{ik}u^{kj}, \quad i \neq j \neq k \neq i$$

Alias: Darboux–Zakharov–Manakov system

➢ The system is the consistency condition of the linear equations

$$\psi^i_{x_j} = u^{ij}\psi^j, \quad i \neq j.$$

The flows D_{x_k}, D_{x_m} commute: $u^{ij}_{x_k,x_m} = u^{ij}_{x_m,x_k}$.

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39 Darboux system discrete

$$u_k^{ij} = (u^{ij} + u^{ik}u^{kj})(I - u^{jk}u^{kj})^{-1}, \quad u^{ij} \in \mathrm{Mat}(N,N), \quad i \neq j \neq k \neq i$$

Alias: discrete Darboux–Zakharov–Manakov system

➢ The system is the consistency condition of the linear equations

$$\psi_j^i = \psi^i - u^{ij}\psi^j, \quad i \neq j.$$

It satisfies the 4D-consistency property $u_{k,m}^{i,j} = u_{m,k}^{i,j}$.

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40 Davey–Stewartson system

$$u_{t_+} = u_{xx} + 2p_x u, \quad -v_{t_+} = v_{xx} + 2p_x v, \quad p_y = uv$$
$$u_{t_-} = u_{yy} + 2q_y u, \quad -v_{t_-} = v_{yy} + 2q_y v, \quad q_x = uv$$

➢ Derived in [1] by multiscale analysis of modulated nonlinear surface gravity waves propagating over a horizontal sea bed. DS system is a two-dimensional analog of nonlinear Schrödinger equation.

➢ The flows ∂_{t_+}, ∂_{t_-} commute. Any linear combination $\alpha\partial_{t_+} + \beta\partial_{t_-}$ is called DS system as well.

➢ The auxiliary linear problems [2, 3]:

$$\begin{cases} \psi_y = u\phi \\ \phi_x = -v\psi \end{cases} \qquad \begin{cases} \psi_{t_+} = \psi_{xx} + 2p_x\psi \\ \phi_{t_+} = v_x\psi - v\psi_x \end{cases} \qquad \begin{cases} -\psi_{t_-} = u\phi_y - u_y\phi \\ -\phi_{t_+} = \phi_{yy} + 2q_y\phi \end{cases}$$

➢ Gauge equivalent systems are the Ishimori equation and the 2D Reyman system.

➢ Third order symmetry:

$$\begin{aligned} u_{t_3} &= u_{xxx} + 3u_x D_y^{-1}(uv)_x + 3uD_y^{-1}(u_x v)_x, \\ v_{t_3} &= v_{xxx} + 3v_x D_y^{-1}(uv)_x + 3vD_y^{-1}(uv_x)_x. \end{aligned} \tag{1}$$

It admits reductions $v = 1$ to the Veselov–Novikov equation and $u = v$ to the modified Veselov–Novikov equation.

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41 Davey–Stewartson system matrix

$$u_t = u_{xx} + 2wu, \quad -v_t = v_{xx} + 2vw, \quad w_y = (uv)_x, \qquad u, v^{\mathsf{T}} \in \mathrm{Mat}(m,n), \quad w \in \mathrm{Mat}(m,m)$$

➢ This and some other analogous examples were introduced in [1].

➢ The linear problem ($\psi \in \mathbb{R}^m$, $\phi \in \mathbb{R}^n$):

$$\psi_y = u\phi, \quad \phi_x = -v\psi, \quad \psi_t = \psi_{xx} + 2w\psi, \quad \phi_t = v_x\psi - v\psi_x.$$

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42 Degasperis–Procesi equation

$$u_t - u_{xxt} = uu_{xxx} + 3u_x u_{xx} - 4uu_x \tag{1}$$

It was shown in [2] that equation

$$u_t - u_{xxt} = uu_{xxx} + bu_x u_{xx} - (b+1)uu_x$$

is integrable only at $b = 2$ (Camassa–Holm equation) or $b = 3$ (1). Although these equations look very similar, the corresponding linear problems are quite different: Camassa–Holm equation is related to the Schrödinger spectral problem which is of the 2nd order, while (1) corresponds to the 3rd order Kaup–Kupershmidt spectral problem

$$\psi_{XXX} + 4V\psi_X + (2V_X - \lambda)\psi = 0, \quad \lambda\psi_T = -p^2\psi_{XX} + pp_X\psi_X + \big(pp_{XX} - p_X^2 + \frac{2}{3}\big)\psi.$$

The compatibility condition is

$$(p^{-1})_T + \big(p(\log p)_{XT} + p^3\big)_X = 0, \quad 2pp_{XX} - p_X^2 + 4Vp^2 + 1 = 0$$

which is equivalent to (1) via the point transformation

$$p^3 = u_{xx} - u, \quad dX = p\,dx - pu\,dt, \quad dT = dt.$$

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43 Differential and pseudo-differential operators

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The role of the differential operators is explained by the fact that the construction problems of finite-gap potentials and higher symmetries of integrable equations are formulated on this language. In both cases, the introducing of pseudo-differential operators is useful. In particular, this is important in the theories of recursion operators and formal symmetry.

1. The problem on commuting differential operators

Multiplication in the ring $\mathcal{R}$ of the ***differential operators (DO)***

$$A = \sum_{k=0}^{n} a_{n-k} D^k = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n, \quad D \equiv \frac{d}{dx}$$

with smooth coefficients $a_k = a_k(x)$ is defined by the ***Leibniz rule***

$$D^m a = aD^m + m a_x D^{m-1} + \frac{m(m-1)}{2} a_{xx} D^{m-2} + \dots.$$

If $A = a_0 D^n + \dots$ and $B = b_0 D^m + \dots$ then

$$[A, B] := AB - BA = (n a_0 b_{0,x} - m b_0 a_{0,x}) D^{n+m-1} + \dots \tag{1}$$

so that, generally, the order of commutator is $n + m - 1$. Therefore, the commutativity condition $[A, B] = 0$ is equivalent to a system of $n + m$ equations for $n + m + 2$ coefficients of A and B. The numbers of equations and unknowns become balanced if we introduce two basic transformations as follows

$$D = a\hat{D}, \qquad \tilde{A} = f^{-1} A f. \tag{2}$$

The first transformation corresponds to the change of the independent variable $x \to \hat{x}$ and the second one is the conjugation with the zero order operator of multiplication by a smooth function $f = f(x)$. Both transformations preserve the property of commutativity. For example, in the case of the conjugation $\widetilde{AB} = \tilde{A}\tilde{B}$ and thus

$$[A, B] = 0 \quad \Leftrightarrow \quad [\tilde{A}, \tilde{B}] = 0.$$

The change of independent variable with $a = a_0^{1/n}$ replaces A by the operator $\hat{A}$ with the leading coefficient $\hat{a}_0 = 1$.

Definition 1. ***Centralizer*** $\mathcal{C}(A)$ of a DO A is the subring of DOs commuting with A:

$$\mathcal{C}(A) = \{B \in \mathcal{R} : \ [A, B] = 0\}.$$

Centralizer is called ***trivial*** if it consists of polynomials with constant coefficients in some minimal order differential operator C, that is

$$A = \alpha_0 C^n + \alpha_1 C^{n-1} + \dots + \alpha_n, \quad B = \beta_0 C^m + \beta_1 C^{m-1} + \dots + \beta_m, \quad \alpha_i = \mathrm{const}, \quad \beta_i = \mathrm{const}\,.$$

It is easy to prove that the centralizer of a first order DO is always trivial. Indeed, if $A = a_0 D + a_1$ then transformations (2) allow to reduce it to $A = D$. Since

$$[D, b_0 D^m + b_1 D^{m-1} + \dots + b_m] = D(b_0) D^m + D(b_1) D^{m-1} + \dots + D(b_m),$$

hence all b_i are constant. Thus, in nontrivial cases the order n of the operator A must be at least 2. In the example below $n = 2$ and order of B is chosen minimal as well.

Example 2. Let $A = D^2 + a$ and $B = D^3 + bD + c$. Then the equation $[A, B] = 0$ is equivalent to the system

$$2b_x = 3a_x, \quad b_{xx} + 2c_x = 3a_{xx}, \quad a_{xxx} + ba_x - c_{xx} = 0.$$

The elimination of b and c yields the equation (ε is an integration constant)

$$a_{xxx} + 6aa_x = \varepsilon a_x, \tag{3}$$

Any solution of this equation gives rise to a commuting pair of DOs. Moreover, it is easy to check that if $u \neq \mathrm{const}$ then no first order operator C exists such that $A = \alpha_0 C^2 + \alpha_1 C + \alpha_2$, so that this pair is not trivial. Particularly, the choice $u = 2x^{-2}$ yields the pair

$$A = D^2 - 2x^{-2}, \quad B = D^3 - 3x^{-2}D + 3x^{-3}, \quad [A, B] = 0, \quad A^3 = B^2.$$

2. The field of pseudo-differential operators

In order to understand the structure of nontrivial centralizers we need to extend the ring $\mathcal{R}$ introducing the ***pseudo-differential operators (PDO)*** as the formal series

$$A = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots \tag{4}$$

The product in the extended ring $\tilde{\mathcal{R}}$ is defined by the ***Leibniz rule*** generalized for any integer power of D:

$$D^n a = \sum_{k=0}^{\infty} \binom{n}{k} D^k(a) D^{n-k} = \begin{cases} \quad \dots & \\ aD^{-1} - a_x D^{-2} + a_{xx} D^{-3} - \dots & n = -1 \\ aD^{-2} - 2a_x D^{-3} + 3a_{xx} D^{-4} - \dots & n = -2 \\ aD^{-3} - 3a_x D^{-4} + 6a_{xx} D^{-5} - \dots & n = -3 \\ \quad \dots & \end{cases}$$

where $\binom{n}{k} = n(n-1)\cdots(n-k+1)/k!$. In particular, for the first order PDO with unit leading coefficient $B = D + b_1 + b_2 D^{-1} + b_3 D^{-2} + \dots$ we find

$$\begin{gathered} B^n = D^n + b_{1,n} D^{n-1} + b_{2,n} D^{n-2} + b_{3,n} D^{n-3} + \dots, \qquad b_{1,n} = n b_1, \\ b_{2,n} = n b_2 + \binom{n}{2}(b_{1,x} + b_1^2), \quad b_{3,n} = n b_3 + \binom{n}{2}(b_{2,x} + 2b_1 b_2) + \binom{n}{3}(b_{1,xx} + 3b_1 b_{1,x} + b_1^3), \ \dots \end{gathered} \tag{5}$$

Therefore the expressions for the coefficients $b_{j,n}$, $j = 1, 2, \dots$ contain only first j coefficients of the given series B. This triangular structure of the equations allows to introduce additional algebraic operations in $\tilde{\mathcal{R}}$.

Lemma 3. *Let A be a formal series (4) of order n with $a_0 = 1$. Then:*
the unique formal series $L = A^{-1}$ exists such that $AL = LA = 1$;
if $n \neq 0$ then the unique formal series $B = A^{1/n}$ exists such that $\operatorname{ord} B = 1$, the leading coefficient is 1 and $B^n = A$.

Proof. The proof is analogous in both cases and we consider only the second one. Starting with formulas (5) we have

$$b_{j,n} = nb_j + f[b_1, b_2, \dots, b_{j-1}]$$

where f is a differential polynomial in its arguments. The system for the coefficients b_k

$$a_1 = nb_1, \quad a_2 = b_{2,n}, \quad a_3 = b_{3,n}, \dots$$

is triangular and is solved uniquely. ■

Two series defined in Lemma are called ***inverse*** and ***n-th root*** correspondingly. The condition $a_0 = 1$ is a technical one and the transformation $D \to a\hat{D}$ (see (2)) with $a = a_0^{1/n}$ leads to a series $\hat{A}$ with unitary leading coefficient.

We would like to stress again that due to triangular structure of equations the first j coefficients of the original series A define first j coefficients of series A^{-1} and $A^{1/n}$. This *recursive property* of algebraic operations in the field $\tilde{\mathcal{R}}$ of power series (4) appears to be very important.

3. Burchnal–Chaundy theorem

Now we can return to the commutativity problem. Consider the centralizer in the ring $\tilde{\mathcal{R}}$:

$$\tilde{\mathcal{C}}(A) = \{B \in \tilde{\mathcal{R}} : \ [A, B] = 0\}.$$

The following statement shows that, in contrast to the case of DOs, this centralizer is always trivial.

Theorem 4 (Burchnal, Chaundy [1])**.** *Let $A \in \tilde{\mathcal{R}}$, $\operatorname{ord} A = n \neq 0$. Then the PDO $B \in \tilde{\mathcal{R}}$ commutes with A if an only if it can be represented as the formal series*

$$B = \beta_0 A_1^m + \beta_1 A_1^{m-1} + \dots, \quad \beta_k = \text{const}, \quad A_1^n = A. \tag{6}$$

Proof. Obviously, any power of A_1 commute with A and belongs to $\tilde{\mathcal{C}}(A)$. In order to prove the opposite statement denote $B_1 = [B, A_1]$. Then

$$BA - AB = BA_1^n - A_1^n B = B_1 A_1^{n-1} + A_1 B_1 A_1^{n-2} + \cdots + A_1^{n-1} B_1 \quad \Rightarrow \quad \tilde{\mathcal{C}}(A_1) = \tilde{\mathcal{C}}(A), \tag{7}$$

since all n terms in the sum (7) have the same leading coefficient.

The leading coefficient b_0 of the PDO $B \in \tilde{\mathcal{C}}(A_1)$ of order $m \neq 0$ must be proportional to a_0^m, due to the formulae (1) which remains valid in $\tilde{\mathcal{R}}$. Therefore, we find that

$$B \in \tilde{\mathcal{C}}(A_1) \quad \Rightarrow \quad b_0 = \beta_0 a_0^m \quad \Rightarrow \quad \tilde{B} = B - \beta_0 A_1^m \in \tilde{\mathcal{C}}(A_1).$$

In order to finish the proof of we use the induction with respect to the order $\tilde{m} < m$ of the series $\tilde{B} = B - \beta_0 A_1^m$. It remains to notice that in the case of order $m = 0$ with $B = b_0 + b_1 D^{-1}$ the formula (1) implies that $a_0 b_{0,x} = 0$. Thus in this case the series $\tilde{B} = B - b_0$ has negative order and the inductive process meets no obstacles. ■

It follows from the above theorem that any centralizer $\tilde{\mathcal{C}}(A)$ is abelian, that is

$$B_1, B_2 \in \tilde{\mathcal{C}}(A) \quad \Rightarrow \quad [B_1, B_2] = 0.$$

In the case of a differential operator A the centralizer $\mathcal{C}(A) \subset \tilde{\mathcal{C}}(A)$ and thus, we obtain a classical result as follows.

Corollary 5. *Any two differential operators commuting with a third one commute with each other.*

In other words, the binary relation $[A, B] = 0$ is an equivalence and one can replace the operator A in $\mathcal{C}(A)$ by any nontrivial (of non zero order) element of the centralizer. The minimal order $n > 0$ of nontrivial elements of $\mathcal{C}(A)$ gives some indication on what is the structure of the centralizer. For $n = 1$ the centralizer is always trivial, but in the case $n = 2$ there are nontrivial ones with odd order differential operators $B \in \mathcal{C}(A)$ (see Example 2). Moreover, although an element $B \in \mathcal{C}(A)$ generally cannot be represented as polynomial in A yet there exist an algebraic relation between A and B in virtue of Theorem 4.

Example 6. Let us consider tersely the structure of $\mathcal{C}(A)$ in the case of second order differential operator A. If the centralizer is nontrivial then it contains a differential operator B_1 of a minimal odd order $2n+1 \geq 3$. Any element in $\mathcal{C}(A)$ can be represented as $P(A)B+Q(A)$ where P, Q are polynomials with constant coefficients. In particular, $B_1^2 = P(A)B_1+Q(A)$ and replacing $B_1 = B+\frac{1}{2}P(A)$ we come to algebraic relation $B^2 = Q(A)$. It is easy to see that the operator B is also of the minimal order $2n+1$, and this is also the degree of the polynomial Q. The relation $B^2 = Q(A)$ completely defines the multiplication in the commutative ring $\mathcal{C}(A)$ with the generators A and B.

4. Residues

Due to Theorem 4 the structure of centralizer $\mathcal{C}(A)$ is related with the properties of the formal series

$$A_1 = a_0 D + a_1 + a_2 D^{-1} + a_3 D^{-2} + \dots, \quad A_1^n = A \in \mathcal{R}. \tag{8}$$

For any PDO

$$B = b_0 D^n + b_1 D^{n-1} + \dots + b_n + b_{n+1} D^{-1} + \dots \quad \in \tilde{\mathcal{R}}$$

we define the ***differential part*** and ***residue***

$$B_+ := b_0 D^n + b_1 D^{n-1} + \dots + b_n \quad \in \mathcal{R}, \qquad \operatorname{res}(B) := b_{n+1}. \tag{9}$$

Particularly, for the formal series (8) we denote

$$\rho_j = \operatorname{res} A_1^j, \quad j = -1, 1, 2, \dots, \quad \rho_0 = a_1/a_0. \tag{10}$$

Inverse above formulae one finds (see (5)) that

$$a_0 = 1/\rho_{-1}, \quad a_1 = \rho_0/\rho_{-1}, \quad a_2 = \rho_1, \quad 2a_3 = \rho_3\rho_{-1} - 2\rho_1\rho_0 - \frac{(\rho_1\rho_{-1})_x}{\rho_{-1}}, \dots$$

and the recursive properties of algebraic operations in $\tilde{\mathcal{R}}$ allows to prove easily (see [2]) the following Lemma.

Lemma 7. *The sequence (10) of the residues of powers of A_1 and the sequence of coefficients of this formal series defines each other uniquely and recursively.*

Definition 8. For a differential operator $L = D^m + l_2 D^{m-2} + \cdots + l_m$ of the special form we call by ***L-hierarchy*** the sequence (10) of residues $\rho_j = \rho_j(L)$, $j \geq 1$ expressed in terms of coefficients $l_2, \ldots, l_m$ of the differential operator L.

Example 9. In the case of the second order DO $L = D^2 + a$ the coefficients of the series $A \in \tilde{\mathcal{R}}$, $A^2 = L$ are expressed through a:

$$\begin{gathered} A = D + a_1 D^{-1} + a_2 D^{-2} + \ldots, \quad 2a_1 = a, \quad 4a_2 = -a_x, \quad 8a_3 = a_{xx} - a^2, \\ 16a_4 = -a_{xxx} + 6aa_x, \quad 2^5 a_5 = a_{xxxx} + 2a^3 - 14aa_{xx} - 11a_x^2, \ldots \end{gathered} \tag{11}$$

That gives for residues $\rho_j(a) = \rho_j(L)$ with odd $j = 1, 3, 5, \ldots$

$$2\rho_1(a) = a, \quad 2^3\rho_3(a) = a_{xx} + 3a^2, \quad 2^5\rho_5(a) = a_{xxxx} + 5a_x^2 + 10aa_{xx} + 10a^3, \ldots \tag{12}$$

All even residues vanish $\rho_{2n}(a) = 0$, $n = 1, 2, 3, \ldots$ because for even powers of series (11) $A^{2n} = L^n$.

One finds by substitution of the expansion (9) into the formula $[L, A^j] = 0$:

$$A^j = (A^j)_+ + \rho_j D^{-1} + \mathcal{O}(D^{-2}) \quad \Rightarrow \quad [L, (A^j)_+] = -2\rho_{j,x}, \tag{13}$$

since (cf (1))

$$0 = [L, A^j] = [L, (A^j)_+] + [L, \rho_j D^{-1}] + \mathcal{O}(D^{-1}) = [L, (A^j)_+] + 2\rho_{j,x} + \mathcal{O}(D^{-1}).$$

Summing up, we see that above definition of L-hierarchy together with Theorem 4 and formula (13) allow to formulate a criterium of non-triviality of the centralizer $\mathcal{C}(L)$ of a differential operator $L = D^2 + a$ of the second order.

Corollary 10 (of Theorem 4)**.** *The centralizer* $\mathcal{C}(D^2 + a)$ *is non-trivial if and only if it contains a differential operator* B *of odd order* $2n+1$, $n \geq 1$ *and in this case the function* $a = a(x)$ *satisfies the nonlinear ODE of order* $2n+1$:

$$\rho_{2n+1}(a) + \sum_{k=0}^{n-1} c_k \rho_{2k+1}(a) = c_n, \quad c_j = \text{const} \in \mathbb{C}.$$

In particular, at $n=1$ the latter equation reads $a_{xx}+3a^2+c_0a=c_1$ (see (12)). It defines the condition of commutativity $[L,(A^3)_+]=0$ and is equivalent to equation (3) from the Example 2.

In conclusion let us discuss briefly the L-hierarchy in the case of the third order operator L. In analogy with Example 9, one finds

$$L=D^3+3uD+3v \quad\Rightarrow\quad A=D+a_1D^{-1}+a_2D^{-2}+\dots,\quad a_1=u,\quad a_2=v-u_x,\dots \tag{14}$$

Like (13), the equality $[L,A^j]=0$ yields

$$A^j=(A^j)_++a_{j,1}D^{-1}+a_{j,2}D^{-2}+\mathcal{O}(D^{-3}) \quad\Rightarrow\quad [L,(A^j)_+]+3a_{j,1,x}D+3a_{j,1,xx}+3a_{j,2,x}=0$$

that is the pair of equations $a_{j,1,x}=a_{j,2,x}=0$. Thus, comparing with (13) this formula includes $a_{j,2,x}$ which should be expressed in terms of $a_{j,1}=\rho_j$. In order to do this we have to suppose additionally that the third order operator L is skew-symmetric:

$$L^\intercal:=-D^3-3Du+3v=-D^3-3uD+3(v-u_x)=-L=-D^3-3uD-3v.$$

In this case

$$A^\intercal+A=0,\quad A^\intercal:=-D-D^{-1}a_1+D^{-2}a_2-D^{-3}a_3+\dots \tag{15}$$

and we get the following lemma.

Lemma 11. *Let, generally,* $A=D+a_1D^{-1}+a_2D^{-2}+\dots$ *and* $A^n=(A^n)_++a_{n,1}D^{-1}+a_{n,2}D^{-2}+\dots$. *Then A is skew-symmetric if and only if* $a_{n,1}=\operatorname{res}(A^n)=0$ *for even* $n=2,4,\dots$. *Moreover, for skew-symmetric A*

$$2a_{n,2}=a_{n,1,x},\quad \textit{if } n \textit{ is odd.}$$

Thus, as well as in the case of symmetric second order differential operators for skew-symmetric third order operators and odd $j=5,7,\dots$

$$a_{j,1,x}=\rho_{j,x}=0 \quad\Rightarrow\quad [L,(A^j)_+]=0.$$

Particularly, in the simplest case $j=5$ an analog of the equation (3) arises

$$u_{xxxxx}+15D\Big(2uu_{xx}+\frac{7}{4}u_x^2+u^3\Big)=\varepsilon u_x. \tag{16}$$

The calculation of $\operatorname{res}(A^5)$ is long enough and, as a matter of fact, it is comparable with straightforward computation of the commutator $[L, M] = a_1 D + a_2$ where

$$L = D^3 + 3uD + 3v, \quad M = (A^5)_+ = D^5 + 5uD^2 + aD + b + c.$$

These computations give firstly

$$a = 5(u_x + v), \quad 3b = 10u_{xx} + 15u^2 + 15v_x, \quad 3c = 10v_{xx} + 30uv$$

and the condition $[L, M] = 0$ results now in the pair of equations

$$\begin{cases} u_{xxxxx} + 15D(uu_{xx} + u_x^2 + 3vu_x - 3v^2 + u^3) = \varepsilon u_x, \\ v_{xxxxx} + 15D(uv_{xx} + 2vu_{xx} + 2v_x u_x - 3vv_x + 3u^2 v) = \varepsilon v_x. \end{cases} \tag{17}$$

It is easy to see that the fifth order equation (16) for u represents one of three possible scalar reductions $\delta v = u_x$, $\delta = 0, 1, 2$ of this system.

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44 Differential substitutions

The formulae (186.2), (186.3) define the prolongation also for transformations more general than the point ones:

$$\tilde{x}_i = f_i(x, u_s), \quad \tilde{u}^j = g^j(x, u_s), \quad |s| \le k. \tag{1}$$

Theorem 1 (Bäcklund [1, 2])**.** *Let the prolongation of the transformation (1) be invertible on some J^r. Then it is point if $m > 1$ or contact if $m = 1$.*

The transformations (1) which are not point or contact are called ***differential substitutions***. It should be stressed that Bäcklund theorem does not mean that any such transformation is not invertible. For example, the following transformation is involutive:

$$\tilde{x} = x, \quad \tilde{u} = \frac{v_x}{u_x}, \quad \tilde{v} = -v + u\frac{v_x}{u_x}.$$

However, it is easy to see that its prolongation does not define an invertible transformation of J^r, for any finite r.

Examples: see substitutions for the Bogoyavlensky–Narita lattices and for the KdV-type equations.

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45 Discrete differential geometry

This field deals with discretization of several notions first discovered within the framework of classical differential geometry in the beginning of the 20-th century [1, 2, 3, 4, 5]. These include many special classes of surfaces and coordinate systems, such as minimal surfaces, surfaces with constant mean curvature, isothermic surfaces, orthogonal and conjugate coordinate systems and more, and also transformations of these objects.

Understanding of classical results from the point of view of modern theory of integrability became possible, in particular, due to the progress in construction of their discrete analogues. The objects of discrete differential geometry are discrete nets, that is, mappings from $\mathbb{Z}^M$ into $\mathbb{R}^d$ (or some other suitable space) specified by certain geometric properties. Their study was initiated in [6, 7]. More recently, the key observation was made that discretization can be defined in terms of Bäcklund–Darboux type transformations and Bianchi permutability property for the continuous objects. On the other hand, continuous objects can be reproduced from the discrete ones under a suitable limit. In many aspects, the discrete picture turns out to be more transparent and fundamental than the continuous one since the transformations of discrete surfaces are described by the same equations as surfaces themselves. This scheme was realized in various settings in the papers [8, 9, 10, 11], see also the book [12] for more details and further references.

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46 Discrete equations

2D discrete equations are those with independent variables on the lattice $\mathbb{Z}^2$.

➢ ***Quad-equations*** are equations of the form

$$Q_{m,n}(u_{m,n}, u_{m+1,n}, u_{m,n+1}, u_{m+1,n+1}) = 0. \tag{1}$$

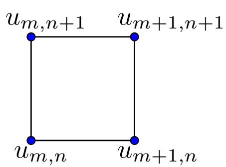
The variables u are associated to the vertices of the square lattice. The equation must be solvable with respect to any of 4 unknowns.

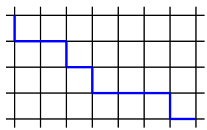
➢ ***Yang–Baxter maps*** are equations of the form

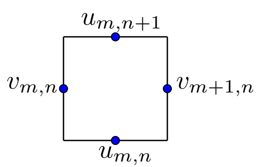
$$u_{m,n+1} = f_{m,n}(u_{m,n}, v_{m,n}), \quad v_{m+1,n} = g_{m,n}(u_{m,n}, v_{m,n}). \tag{2}$$

The dependent variables u, v are associated to the edges of the square lattice.

➢ The simplest choice of the initial data for both types of equations is along the coordinate axes or on the “staircase”.







➢ Discrete Toda type system on a planar graph G is a set of equations of the form

$$\sum_{j:(i,j)\in E_G} f_{i,j}(u_i,u_j)=0,\quad i\in V_G$$

where V_G, E_G are sets of the vertices and the edges of G respectively. In particular, for the cases of square and triangular lattices we obtain two following types of equations.

➢ ***Discrete Toda type lattices*** are equations of the form

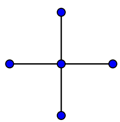
$$f^1_{m,n}(u_{m,n},u_{m-1,n})+f^2_{m,n}(u_{m,n},u_{m+1,n})+f^3_{m,n}(u_{m,n},u_{m,n-1})+f^4_{m,n}(u_{m,n},u_{m,n+1})=0.$$

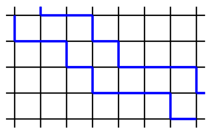
The simplest choice of initial data is on the pair of lines $n=0$, $n=1$.

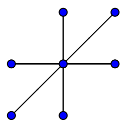
➢ ***Discrete relativistic Toda type lattices***:

$$f^1_{m,n}(u_{m,n},u_{m-1,n})+f^2_{m,n}(u_{m,n},u_{m+1,n})+f^3_{m,n}(u_{m,n},u_{m,n-1})+f^4_{m,n}(u_{m,n},u_{m,n+1})\\ +f^5_{m,n}(u_{m,n},u_{m-1,n-1})+f^6_{m,n}(u_{m,n},u_{m+1,n+1})=0.$$

The simplest choice of initial data is on the double staircase.







47 Dispersion and dissipation

Dispersion is the destruction of wave packets due to the dependence of the wave velocity on the wave vector. ***Dissipation*** is the decay of the wave amplitude at $t \to \infty$. Both phenomena can be explained within the scope of the linear theory of waves, however they play the great role for the waves of nonlinear nature as well.

Any linear partial differential equation with constant coefficients $L[u(t,x)] = 0$ admits solutions in the form of the planar harmonic waves $u(t,x) = \exp(i(\langle k,x\rangle - \omega t))$ where the frequency ω and the wave vector k are related by certain algebraic equation $\Lambda(\omega,k) = 0$ which is called the ***dispersion law***. For example, the direct substitution proves:

wave equation	$u_{tt} = \Delta u$	$\mapsto$	$\omega^2 = \langle k,k\rangle,$
Klein–Gordon equation	$u_{tt} = \Delta u - cu$	$\mapsto$	$\omega^2 = \langle k,k\rangle + c,$
heat equation	$u_t = \Delta u$	$\mapsto$	$\omega = -i\langle k,k\rangle,$
Schrödinger equation	$iu_t = \Delta u$	$\mapsto$	$\omega = -\langle k,k\rangle.$

The hyperplane $\langle k,x\rangle = \omega t + \text{const}$ is called the surface of the constant phase. It propagates along the unit normal vector $k/|k|$ with the ***phase velocity*** $v_p = \omega/|k|$. The dependence of the frequency on the wave vector is characterized by the ***group velocity*** $v_g = \nabla_k(\omega)$. If $v_g \neq \text{const}$ then the different modes propagate with the different velocities and this is why the dispersion takes place.

Dissipation takes place if the frequency has the negative imaginary part: $\omega = \omega_R + i\omega_I$, $\omega_I < 0$, in this case the waves decay exponentially. In contrast, the dispersion law with $\omega_I > 0$ leads to the exponential growth and instability of the waves.

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48 Dispersive long waves system

$$u_t = (u_x + u^2 - 2q)_x, \quad -v_t = (v_x - 2uv)_x, \quad q_y = v_x$$

- Introduced in [1, 2, 3].
- Bäcklund transformation [4, 5]:

$$u_{n,y} = v_n - v_{n+1}, \quad v_{n,x} = v_n(u_n - u_{n-1}).$$

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49 Dispersive water waves system

$$u_t = (u_{xx} - 3vu_x + 3uv^2 - 3u^2)_x, \quad v_t = (v_{xx} + 3vv_x + v^3 - 6uv)_x$$

References

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50 Dressing chain

$$v'_{n+1} + v'_n = (v_{n+1} - v_n)^2 + \beta_n$$

➢ This differential-difference equation defines the Darboux transformation for the Schrödinger equation and x-part of the Bäcklund transformation for the potential KdV equation.

➢ The differences $f_n = v_{n+1} - v_n$ satisfy the equations

$$f'_{n+1} + f'_n = f^2_{n+1} - f^2_n + \beta_{n+1} - \beta_n \tag{1}$$

which corresponds to the modified KdV equation. In this form, the dressing chain appeared in [1] as a tool for solving quantum problems of certain types, see factorization method.

➢ Zero curvature representation: $L'_n = U_{n+1}L_n - L_nU_n$, where

$$U_n = \begin{pmatrix} v_n & 1 \\ v'_n - v^2_n - \lambda & -v_n \end{pmatrix}, \qquad L_n = \begin{pmatrix} v_{n+1} & 1 \\ \beta_n - v_n v_{n+1} - \lambda & -v_n \end{pmatrix}$$

or

$$U_n = \begin{pmatrix} 0 & 1 \\ u_n - \lambda & 0 \end{pmatrix}, \qquad L_n = \begin{pmatrix} f_n & 1 \\ f^2_n + \beta_n - \lambda & f_n \end{pmatrix}, \quad u_n = 2v'_n.$$

➢ In a wide sense, the term ***dressing chain*** is applied to any differential-difference equation generated by Darboux transformations.

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51 Dressing chain, 2-dimensional

$$f_{n,x} + f_{n+1,x} = f_n^2 - f_{n+1}^2 - \sigma(g_n - g_{n+1}), \quad g_{n,x} = f_{n,y}$$

$$(v_n + v_{n+1})_x = (v_n - v_{n+1})^2 - \sigma g_n, \quad g_{n,x} = (v_n - v_{n+1})_y.$$

52 Dressing chain, matrix

$$v'_{n+1} + v'_n = (v_{n+1} - v_n)^2 + b_n, \quad b'_n = [b_n, v_{n+1} - v_n], \quad v_n, b_n \in \mathrm{Mat}_n$$

The differences $f_n = v_{n+1} - v_n$ satisfy the equations

$$f'_{n+1} + f'_n = f^2_{n+1} - f^2_n + b_{n+1} - b_n, \quad b'_n = [b_n, f_n]$$

➢ Zero curvature representation: $L'_n = U_{n+1}L_n - L_nU_n$, where

$$U_n = \begin{pmatrix} v_n & I \\ v'_n - v^2_n - \lambda I & -v_n \end{pmatrix}, \qquad L_n = \begin{pmatrix} v_{n+1} & I \\ b_n - v_n v_{n+1} - \lambda I & -v_n \end{pmatrix}$$

or

$$U_n = \begin{pmatrix} 0 & I \\ u_n - \lambda I & 0 \end{pmatrix}, \qquad L_n = \begin{pmatrix} f_n & I \\ f^2_n + b_n - \lambda I & f_n \end{pmatrix}, \quad u_n = 2v'_n.$$

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53 Dressing chain, matrix twodimensional

$$f_{n+1,x} + f_{n,x} = f_{n+1}^2 - f_n^2 + p_{n+1} - p_n, \quad p_{n,x} = f_{n,t} + [p_n, f_n].$$

54 Dym equation

$$u_t = u^3 u_{xxx}$$

or $(u = -2^{-1/3} v^{-1/2})$

$$v_t = (v^{-1/2})_{xxx}$$

➢ The equation is related to Schwarzian KdV by the composition of introducing a potential and hodograph transformation:

$$u_t = u^3 u_{xxx} \quad \Leftarrow \quad y_x = \frac{1}{u}, \;\; y_t = \frac{1}{2} u_x^2 - u u_{xx} \quad \Rightarrow \quad y_t = \frac{y_{xxx}}{y_x^3} - \frac{3 y_{xx}^2}{2 y_x^4} \quad \Leftrightarrow \quad x_t = x_{yyy} - \frac{3 x_{yy}^2}{2 x_y}$$

The relation of this substitution with the Liouville transformation for Sturm–Liouville operators is discussed in [2].

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55 Eckhaus equation

$$iu_t = u_{xx} + 2au(|u|^2)_x + |a|^2|u|^4u$$

Linearizable by the substitution [1].

Multifield generalizations were discussed in [2]. A discretization was proposed in [3].

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56 Elliptic functions

Weierstrass functions

$$\sigma(z) = z\prod{}'\Big(1-\frac{z}{w}\Big)\exp\Big(\frac{z}{w}+\frac{z^2}{2w^2}\Big), \qquad \zeta = \frac{\sigma'}{\sigma}$$

$$\zeta(z) = \frac{1}{z}+\sum{}'\Big(\frac{1}{z-w}+\frac{1}{w}+\frac{z}{w^2}\Big), \qquad \zeta' = -\wp$$

$$\wp(z) = \frac{1}{z^2}+\sum{}'\Big(\frac{1}{(z-w)^2}-\frac{1}{w^2}\Big), \qquad (\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp-e_1)(\wp-e_2)(\wp-e_3)$$

where sum and product are over the lattice

$$w = 2m\omega_1+2n\omega_2, \quad m,n\in\mathbb{Z}, \quad \operatorname{Im}\omega_2/\omega_1>0, \quad e_1=\wp(\omega_1), \quad e_2=\wp(\omega_2), \quad e_3=\wp(\omega_1+\omega_2)$$

and prime denotes that the point $w=0$ is excluded.

$$\sigma(-z) = -\sigma(z), \quad \sigma(z+2\omega_j) = -e^{2\eta_j(z+\omega_j)}\sigma(z), \quad j=1,2$$

$$\zeta(-z) = -\zeta(z), \quad \zeta(z+2\omega_j) = \zeta(z)+2\eta_j, \quad \eta_j=\zeta(\omega_j), \quad \eta_1\omega_2-\eta_2\omega_1 = \frac{1}{2}\pi i$$

$$\wp(-z) = \wp(z), \quad \wp(z+2\omega_j) = \wp(z)$$

Any elliptic function $f(z)$ (with the periods w) can be represented by formula

$$f(z) = \text{const}\,\frac{\sigma(z-a_1)\cdots\sigma(z-a_r)}{\sigma(z-b_1)\cdots\sigma(z-b_r)}$$

where a_j, b_j are respectively the zeroes and poles of $f(z)$ in the fundamental parallelogram $\Omega=\{z=t_1\omega_1+t_2\omega_2:\ 0\le t_1,t_2<2\}$.

Several most useful identities:

$$\sigma(x+\alpha)\sigma(x-\alpha)\sigma(\beta+\gamma)\sigma(\beta-\gamma) = \sigma(x+\beta)\sigma(x-\beta)\sigma(\alpha+\gamma)\sigma(\alpha-\gamma) - \sigma(x+\gamma)\sigma(x-\gamma)\sigma(\alpha+\beta)\sigma(\alpha-\beta)$$

$$\zeta(x)+\zeta(y)+\zeta(z)-\zeta(x+y+z) = \frac{\sigma(x+y)\sigma(y+z)\sigma(z+x)}{\sigma(x)\sigma(y)\sigma(z)\sigma(x+y+z)}$$

$$\frac{1}{2}\begin{vmatrix} 1 & \wp(x) & \wp'(x) \\ 1 & \wp(y) & \wp'(y) \\ 1 & \wp(z) & \wp'(z) \end{vmatrix} = \frac{\sigma(x+y+z)\sigma(x-y)\sigma(y-z)\sigma(z-x)}{\sigma^3(x)\sigma^3(y)\sigma^3(z)}$$

$$\wp(x)-\wp(y) = -\frac{\sigma(x+y)\sigma(x-y)}{\sigma^2(x)\sigma^2(y)}$$

$$\frac{1}{4}\Big(\frac{\wp'(x)+\wp'(y)}{\wp(x)-\wp(y)}\Big)^2 = \wp(x)+\wp(y)+\wp(x-y) \qquad (1)$$

The biquadratic polynomial

$$H(u,v,w) = (uv+vw+wu+g_2/4)^2 - (u+v+w)(4uvw-g_3)$$

satisfies the identity $H_v^2 - 2HH_{vv} = r(u)r(w)$, $r(u) = 4u^3 - g_2u - g_3$ (see Möbius invariants). The identity

$$H(\wp(x),\wp(y),\wp(z)) = -\frac{\sigma(x+y+z)\sigma(-x+y+z)\sigma(x-y+z)\sigma(x+y-z)}{\sigma^4(x)\sigma^4(y)\sigma^4(z)} = (\wp(x)-\wp(y))^2(\wp(x+y)-\wp(z))(\wp(x-y)-\wp(z))$$

implies the Euler form of the addition theorem (1) $H(\wp(x),\wp(y),\wp(x\pm y)) = 0$.

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57 Ermakov system

$$\ddot{x} + \omega^2(t)x = x^{-3}f(x/y), \quad \ddot{y} + \omega^2(t)y = y^{-3}g(x/y)$$

➢ The case $f = \text{const}$ was introduced in the paper [1].

➢ In the general case, the system possesses the first integral

$$I = \frac{1}{2}(x\dot{y} - y\dot{x})^2 - \int^{x/y} z^{-3}f(z)dz - \int^{y/z} z^{-3}g(z)dz$$

and is linearizable [2, 3].

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58 Ernst equation

$$\mathrm{Re}(u)\left(u_{xx} + u_{yy} + \frac{u_x}{x}\right) = u_x^2 + u_y^2$$

59 Euler top

$$\dot{M} = [M, \Omega], \quad M = J\Omega + \Omega J, \quad M, \Omega \in \mathrm{so}(d)$$

This ODE describes the rotation of a heavy rigid d-dimensional body around its fixed center of mass. The case $d = 3$ was solved in elliptic functions by Euler. The general case was first considered in [1] where some first integrals were presented. The complete set of the first integrals and the Lax representation

$$\frac{d}{dt}(M + \lambda J^2) = [M + \lambda J^2, \Omega + \lambda J]$$

were found in [2].

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60 Euler top in quadratic potential

$$\dot{M} = [M, \Omega] + [P, J], \quad \dot{P} = [P, \Omega], \quad M = J\Omega + \Omega J, \quad M, \Omega \in \mathrm{so}(d), \quad P = P^{\mathsf{T}}$$

This corresponds to the rotation of a d-dimensional rigid body around its fixed center of mass in the Newtonian gravitational field with an arbitrary quadratic potential. Integrability of this problem was proved in [1, 2, 3].

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61 Euler top discrete

$$M_{n+1} = W_n^{\intercal} M_n W_n, \quad M_n = W_n J - J W_n^{\intercal}, \quad M_n \in \mathrm{so}(d), \quad W_n \in \mathrm{SO}(d)$$

Continuous limit: $W_n = I + \varepsilon\Omega(\varepsilon n) + o(\varepsilon^2)$, $M_n = \varepsilon M(\varepsilon n)$.

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62 Euler top, discrete in quadratic potential

$$M_{n+1} = W_n^{\intercal} M_n W_n + [P_{n+1}, JW_n + W_n^{\intercal} J], \quad P_{n+1} = W_n^{\intercal} P_n W_n,$$

$$M_n = W_n J - J W_n^{\intercal} + \frac{1}{2}(J W_n^{\intercal} P_n - P_n W_n J), \quad M_n \in \mathrm{so}(d), \quad W_n \in \mathrm{SO}(d), \quad P_n = P_n^{\intercal}$$

Continuous limit: $W_n = I + \varepsilon\Omega(\varepsilon n) + o(\varepsilon^2)$, $M_n = \varepsilon M(\varepsilon n)$, $P_n = \varepsilon^2 P(\varepsilon n)$.

References

[1] Yu.B. Suris. The problem of integrable discretization: Hamiltonian approach. Basel: Birkhäuser, 2003.

63 Euler–Darboux equation

Author: V.G. Marikhin, 27.08.2007

$$u_{xy} + \frac{\alpha u_x - \beta u_y}{x - y} = 0 \tag{1}$$

The general "solution" is of the form

$$u = \int \rho(t)(x-t)^{\beta}(y-t)^{\alpha} dt.$$

The Euler–Darboux operator $L = \partial_x\partial_y + \frac{\alpha}{x-y}\partial_x - \frac{\beta}{x-y}\partial_y$ and quantum spin operators

$$S^1 = -\frac{1}{2}(x^2-1)\partial_x - \frac{1}{2}(y^2-1)\partial_y + \frac{1}{2}(\beta x + \alpha y),$$
$$S^2 = -\frac{i}{2}(x^2+1)\partial_x - \frac{i}{2}(y^2+1)\partial_y + \frac{i}{2}(\beta x + \alpha y),$$
$$S^3 = -x\partial_x - y\partial_y + \frac{1}{2}(\alpha+\beta)$$

generate the algebra with identities

$$[S^a, S^b] = i\varepsilon^{abc}S^c, \quad [S^1, L] = (x+y)L, \quad [S^2, L] = i(x+y)L, \quad [S^3, L] = 2L$$
$$S_1^2 + S_2^2 + S_3^2 + (x-y)^2 L = s(s+1), \quad s = \frac{1}{2}(\alpha+\beta).$$

Therefore S^i are the Bäcklund operators of Euler–Darboux equation, that is if u is a solution of (1) then $u^i = S^i u$ are solutions as well. For example, the seed solution $u_0 = (x-y)^{\alpha+\beta+1}$ generates the family of solutions $u_n = P_n(x,y)u_0$. Several first polynomials are

$$P_1 = (\alpha+1)x + (\beta+1)y, \quad P_2 = (\alpha+1)(\alpha+2)x^2 + 2xy(\alpha+1)(\beta+1) + (\beta+1)(\beta+2)y^2,$$
$$P_3 = (\alpha+1)(\alpha+2)(\alpha+3)x^3 + 3(\alpha+1)(\alpha+2)(\beta+1)x^2 y$$
$$+3(\alpha+1)(\beta+1)(\beta+2)xy^2 + (\beta+1)(\beta+2)(\beta+3)y^3.$$

64 Euler–Poisson equations

$$u' = [u, Ju] + [\gamma, v], \quad v' = [v, Ju], \quad u, v, \gamma \in \mathbb{R}^3, \quad J = \operatorname{diag}(J_1, J_2, J_3), \quad \gamma, J = \mathrm{const}$$

The system describes the motion of a rigid body spinning around a fixed point in a uniform gravitational field, in three dimensions. The system is not integrable for the generic values of parameters γ, J. However, several integrable cases are known. Three quadratic integrals of motion exist for any set of parameters:

$$\langle v, v\rangle = 1, \quad \langle u, v\rangle = \sigma, \quad \langle u, Ju\rangle - 2\langle \gamma, v\rangle = \varepsilon.$$

Complete integrability requires one more first integral. It exists in the following cases:

case	parameters	first integral
Euler	$\gamma = 0$	$\langle u, u\rangle$
Lagrange	$J_1 = J_2, \quad \gamma_1 = \gamma_2 = 0$	u_3
Kowalewskaya	$2J_1 = 2J_2 = J_3, \quad \gamma_3 = 0$	$\lvert J_1(u_1 + iu_2)^2 + 2(\gamma_1 + i\gamma_2)(v_1 + iv_2)\rvert^2$

Lagrange case contains, in particular, the isotropic subcase $J = \mathrm{id}$ with the first integral $\langle \gamma, u\rangle$. Kowalevskaya case was discovered under the following setting: to find cases when the general solution of the system is meromorphic in t. Up to the obvious changes, the above list covers all cases which satisfy this property. In contrast to the first two cases with solutions expressed in terms of elliptic functions, the solution of the Kowalevskaya top is given in genus 2 hyperelliptic functions.

Several more cases are partially integrable, that is, are integrable on some invariant level set: the cases of Goryachov, Hess–Appelrot, Bobylev–Steklov and N. Kovalevski.

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65 Evolutionary equations

Evolutionary equations are PDE of the form $\vec{u}_t = F[\vec{x}, \vec{u}]$, where function F depends on partial derivatives $\vec{u}$ (up to some fixed order) with respect to the spatial independet variables $\vec{x}$. Often, the dependence is allowed on the nonlocal variables, that is, the quantities related with $\vec{u}$ by means of some differential constraint. The simplest example of nonlocality gives the expression $D_x^{-1}(u)$ which enters the r.h.s. of KP equation. The classification problem for evolutionary equations with two spatial variables x, y was considered in [1].

➢ The most studied is the theory of scalar local evolutionary equations with one spatial variable:

$$u_t = f(x, u, u_1, \dots, u_n), \quad u_k = D_x^k(u).$$

It can be proved that the even order scalar evolution equations do not possess the higher order conservation laws. Therefore, the order of the canonical densities is bounded above and this essentially simplifies the classification. For the 2-nd order equations, it was obtained in [2] (see Theorem 23.1), 4-th order equations were classified in [3]. It turns out that all these equations are linearizable via differential substitutions. The most known example is the Burgers equation linearizable by the Cole–Hopf transform, and the whole class is often called the Burgers-type equations.

The integrable equations of the odd order are divided into two types. The first one consists of the Burgers-type equations and therefore is not of particular interest. The nature of the equations of the second type is quite different. These are the equations solvable by ISTM, they possess the infinite set of higher conservation laws and their higher symmetries are also of the odd order. Quite naturally, the class

$$u_t = F(u_3, u_2, u_1, u, x, t) \tag{1}$$

containing the famous KdV equation have attracted the attention of many researches. One of the early results was obtained by Ibragimov and Shabat [4] who proved that the integrable equations (1) were divided into the following subclasses:

$$u_t = au_3 + b; \qquad u_t = \frac{1}{(au_3 + b)^2} + c; \qquad u_t = \frac{2au_3 + b}{\sqrt{au_3^2 + bu_3 + c}} + d, \quad b^2 \neq 4ac,$$

where a, b, c, d depend on u_2, u_1, u, x, t. The first classification result, namely, for the equations of the special form

$$u_t = u_3 + f(u_1, u)$$

was obtained in [5, 6]. The complete list of the KdV-type integrable equations with the constant *separant*, that is, of the form

$$u_t = u_3 + f(u_2, u_1, u, x) \tag{2}$$

was presented in [7]. This result was an important step in the development of the symmetry approach. The special quasilinear case

$$u_t = a(u_1, u, t)u_3$$

was classified in [8]. The classification of the general case (1) was initiated in the papers [9, 10, 11, 12], however the full solution of this challenging problem is not obtained so far. Most probably, no essentially new equations can be found in the rest cases, accordingly to the following conjecture.

Conjecture 1 ([10]). *Any integrable equation (1) is related via a contact transform or a differential substitution either to KdV, or to Krichever–Novikov or to the linear equation* $u_t = u_3 + a(x,t)u_1 + b(x,t)u$.

It is also not clear, how many higher symmetries are actually necessary for the integrability of equation (1). It may be possible that the Fokas conjecture is valid for this class of equations, that is the existence of a single 5-th order symmetry implies the integrability.

➢ The integrable equations of the 5-th order were classified only in the constant separant case [13]

$$u_t = u_5 + F(u_4, u_3, u_2, u_1, u).$$

These can be divided into three types: symmetries of the Burgers-type equations, symmetries of the KdV-type equations (2) and the equations without lower-order symmetries. The most known representatives of the latter subclass are the Kaup–Kupershmidt and Sawada–Kotera equations. The equations of this type admit the zero curvature representations in 3×3 matrices, in contrast to the equations (2) for which 2×2 matrices suffice.

➢ Regarding the 7-th and higher order equations only particular results are known [14, 15].

➢ The two-component evolution systems of the form

$$\vec{u}_t = A(\vec{u})\vec{u}_{xx} + F(\vec{u}, \vec{u}_x), \quad \vec{u} = (u, v)$$

where classified in the papers [16, 17, 18]. They are also divided into three types: equations of NLS and Boussinesq type with the zero curvature representations in 2×2 and 3×3 matrices respectively, and linearizable equations.

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66 Equivalence problem

The equivalence problem consists of finding the necessary and sufficient conditions which allow to determine whether two equations of the class under consideration are equivalent modulo some set of transformations, and in the effective construction of such transformation if it exists. As usual, the admissible transformations are assumed to be the point or contact ones or their subgroups preserving the general form of equations under scrutiny; sometimes differential substitutions are allowed as well.

The importance of this problem is explained by the fact that the differential equations are not an invariant object and therefore the study of transformations must be an essential part of the general theory.

The classical work [4] demonstrates the complexity of such sort of the problems even in the simplest case of second order ODE.

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67 Factorization method

The applications of Darboux transformation in quantum mechanics had been stimulated by Schrödinger papers [1, 2, 3] where this method had been applied to the whole range of the problems, such as construction of adjoint spherical harmonics, hypergeometric equation, the Kepler problem in the flat space and on the hypersphere. The detailed summary of the results obtained in this period is given in [4]. Following this paper, let us look for the solutions of the dressing chain (50.1)

$$f'_{n+1}+f'_n=f^2_{n+1}-f^2_n+\beta_{n+1}-\beta_n$$

in the form

$$f_n=(n+c)f+g+h/(n+c) \qquad (1)$$

(If $c\in\mathbb{Z}$ then only one half of the chain is considered, at $n<-c$ or $n>-c$.) This Ansatz reduces the lattice to the system

$$f'+f^2=c_1,\quad g'+fg=c_2,\quad gh=c_3,\quad h^2=c_4,\quad c_i=\text{const},$$
$$-\beta_n=c_1m^2+2c_2m+2c_3/m+c_4/m^2,\quad u_n=-m(m-1)f'-(2m-1)g'+g^2+2fh.$$

The analysis of all possible branches brings to the table below. The answers are simplified, where possible, by the scalings and reflection:

$$\tilde f_n(x)=af_n(ax+b),\quad \tilde\beta_n=a^2\beta_n+\beta,\quad \tilde u_n(x)=a^2u_n(ax+b)+\beta,$$
$$\tilde f_n=-f_{-n},\quad \tilde\beta_n=\beta_{-n},\quad \tilde u_n=u_{-n+1}.$$

Three cases are essentially different for the solutions (A) and (E): $a=1,b=0$; $a=i,b=0$; $a=i,b=\pi/2$.

The very simple formula (1) is remarkable since all solutions lead to potentials which are of interest in quantum mechanics. The first and simplest application was related to the well-known harmonic oscillator, but some of the other potentials were first discovered only by this method.

The abridged classification of factorization types accordingly to Infeld and Hull

	f_n	β_n	u_n	
(A)	$am\cot(ax+b)+\dfrac{d}{\sin(ax+b)}$	a^2m^2	$\dfrac{1}{\sin^2(ax+b)}(a^2m(m-1)+d^2$ $+ad(2m-1)\cos(ax+b))$	[5]
(B)	e^x-m	$-m^2$	$e^{2x}-(2m-1)e^x$	[6]
(C)	$\dfrac{m}{x}+dx$	$-d(4m+1)$	$\dfrac{m(m-1)}{x^2}-2dm+d^2x^2$	
(D)	$-x$	$2n+1$	x^2+2n	
(E)	$am\cot(ax+b)+\dfrac{d}{m}$	$a^2m^2-d^2/m^2$	$m(m-1)\dfrac{a^2}{\sin^2(ax+b)}$ $+2ad\cot(ax+b)$	[7, 8]
(F)	$\dfrac{m}{x}+\dfrac{d}{m}$	$-d^2/m^2$	$\dfrac{m(m-1)}{x^2}+\dfrac{2d}{x}$	

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68 Fermi–Pasta–Ulam–Tsingou lattice

Author: V.E. Adler, 26.12.2008

$$\omega^{-2}u_{n,tt} = u_{n+1} - 2u_n + u_{n-1} + a(u_{n+1} - u_n)^2 - a(u_n - u_{n-1})^2 \quad (1)$$

The system describes an one-dimensional lattice of anharmonic oscillators. Its numeric investigation was undertaken in the paper [1]. It was expected that the nonlinear interaction would result quickly in an uniform distribution of the energy over all modes, in accordance with Debye theory [2], however it turned out that the energy transport occured only between few lower modes. Due to the periodic boundary conditions $u_n = u_{n+N}$ the recurrence of initial states was observed. (The capacity of MANIAC-I, the first computer in the world, which was used in this first numerical experiment in mathematical physics allowed to take $N = 64$.) A qualitative explanation of recurrence phenomena was proposed in [3] on the base of the notion of solitons, that is the nonlinear travelling waves which interact elastically with each other. More precisely, this notion was introduced not for the lattice (1), but for Korteweg-de Vries equation which is its continuous limit. In turn, KdV equation was replaced, in the numeric study, by the difference equation

$$u_n^{j+1} = u_n^{j-1} - \frac{k}{3h}(u_{n+1}^j + u_n^j + u_{n-1}^j)(u_{n+1}^j - u_{n-1}^j) - \frac{\delta^2 k}{h^3}(u_{n+2}^j - 2u_{n+1}^j + 2u_{n-1}^j - u_{n+2}^j)$$

with the periodicity condition $u_n^j = u_{n+2N}^j$. It should be stressed that both this discretization and the lattice (1) itself are not integrable, so that, strictly speaking, the waves in both numerical experiments demonstrated only soliton-like behavior. Nevertheless, the further studies lead to the development of the exact theory of soliton solutions and to discovery of the integration method of KdV by means of the inverse scattering problem. The complete explanation of the recurrence phenomenon was obtained after development of the theory of finite-gap solutions (the soliton solutions correspond to the limit $N \to \infty$).

➢ The continuous limit for the lattice (1): let $u_n(t) = u(x,\tau)$, $x = nh$, $\tau = \omega ht$, then Taylor expansion of $u_{n\pm1}$ is

$$u_{n\pm1} = u \pm hu_x + \frac{h^2}{2}u_{xx} \pm \frac{h^3}{6}u_{xxx} + \frac{h^4}{24}u_{xxxx} + o(h^5), \quad h \to 0,$$

and one comes to the Boussinesq-type approximation

$$u_{\tau\tau} = u_{xx} + 2ahu_x u_{xx} + \frac{h^2}{12} u_{xxxx} + o(h^3),$$

which describes the wave propagation in both directions. Next, assume that the parameter a is although small, $a = \kappa h$. Then the change $u(x,\tau) = v(X,T)$, $X = x + \tau$, $T = \kappa h^2 \tau$, $24\kappa = \delta^{-1}$ brings to

$$V_{XT} = V_X V_{XX} + \delta V_{XXXX} + o(h),$$

that is the KdV equation for V_X.

➢ The detailed discussion of FPU experiment is given in the books [4, 5, 6]. The preprint [1] was reprinted in number of sources, their list and some interesting, but not well-known historical facts can be found in [7]. In should be mentioned that a lattice, analogous to (1) was proposed earlier in the paper [8].

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69 Fischer equation

$$u_t = u_{xx} + u - u^2$$

➢ Applications in biology and chemical kinetics.

➢ Not integrable [3, 4]. Some exact solutions are found in [5].

➢ See also: Burgers–Huxley, Kolmogorov–Petrovsky–Piskunov equations.

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70 Fornberg–Whitham equation

$$u_t - u_{xxt} + u_x = uu_{xxx} + 3u_x u_{xx} - uu_x$$

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71 Frenkel–Kontorova lattice

$$u_{n,tt} = \gamma(u_{n+1} - 2u_n + u_{n-1}) - \sin u_n$$

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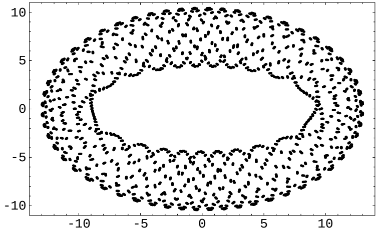
72 Garnier system

$$u'' = \langle u, v\rangle u + Ju, \quad v'' = \langle u, v\rangle v + Jv, \quad u, v \in \mathbb{R}^d, \quad J = \mathrm{diag}(J_1, \dots, J_d)$$

73 Garnier system discrete

$$\langle u_{n+1}, u_n \rangle u_{n+1} + \langle u_n, u_n \rangle u_n + \langle u_n, u_{n-1} \rangle u_{n-1} = J u_n, \quad u_n \in \mathbb{R}^d, \quad J = \mathrm{diag}(J_1, \ldots, J_d)$$



74 Gauge transformations

Example 1. The ***Liouville transformation***

$$dx = r^2 dy, \quad \psi = r\phi, \quad u = q/r^4 + r_{xx}/r$$

relates two forms of Sturm–Liouville equation:

$$\psi_{xx} = (u(x) - \lambda)\psi \quad \leftrightarrow \quad \phi_{yy} = (q(y) - \lambda r^4(y))\phi.$$

In particular, the choice $r = \psi(x, 0)$ brings to another canonical form (so called ***acoustic spectral problem***)

$$\phi_{yy} = -\lambda r^4(y)\phi. \tag{1}$$

For this, the Darboux transformation (37.1) is gauge invariant to the following one [1]:

$$\hat{\phi} = \phi_y/p - \phi, \quad p := \phi_y^{(\alpha)}/\phi^{(\alpha)}, \quad p_y + p^2 = -\alpha r^4, \quad \hat{r} = p/r, \quad \hat{r}^2 d\hat{y} = r^2 dy.$$

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75 Gerdjikov–Ivanov equation

$$iu_t = u_{xx} - iu^2\bar{u}_x + \frac{1}{2}u^3\bar{u}^2$$

Alias: DNLS-III equation

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76 Hamiltonian structure

Author: A.Ya. Maltsev, 2.10.2009

General references: [1, 2, 3, 4, 5]

1. Finite-dimensional dynamical systems

A ***Poisson bracket*** on a finite-dimensional manifold $\mathcal{M}^n$ with local coordinates $\mathbf{x} = (x^1, \ldots, x^n)$ is given by a contravariant 2-tensor $J^{ij}(\mathbf{x})$ which is skew-symmetric

$$J^{ij}(\mathbf{x}) = -J^{ji}(\mathbf{x}) \tag{1}$$

and satisfies the ***Jacobi identity***

$$J^{iq} \frac{\partial J^{jk}}{\partial x^q} + J^{jq} \frac{\partial J^{ki}}{\partial x^q} + J^{kq} \frac{\partial J^{ij}}{\partial x^q} = 0 \tag{2}$$

(summation over repeated indices is assumed everywhere). The Poisson bracket of two smooth functions $f(\mathbf{x})$, $g(\mathbf{x})$ on $\mathcal{M}^n$ is given then by the formula

$$\{f, g\} = \frac{\partial f}{\partial x^i} J^{ij} \frac{\partial g}{\partial x^j}.$$

It is easy to see that $\{x^i, x^j\} = J^{ij}(\mathbf{x})$.

The following identities take place on the space of smooth functions on $\mathcal{M}^n$:

bilinearity $\quad \{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\},$

skew-symmetry	$\{f, g\} = -\{g, f\}$,
Leibnitz identity	$\{fh, g\} = f\{h, g\} + h\{f, g\}$,
Jacobi identity	$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$

Thus, a Poisson bracket gives a structure of Lie algebra on the space of smooth functions on $\mathcal{M}^n$. The Poisson bracket on $\mathcal{M}^n$ is called also a ***Poisson structure*** on $\mathcal{M}^n$ and the manifold $\mathcal{M}^n$ is called a ***Poisson manifold*** in this case.

A Poisson bracket on $\mathcal{M}^n$ is called non-degenerate if $\det ||J^{ij}|| \neq 0$ everywhere on $\mathcal{M}^n$. A non-degenerate Poisson bracket can exist only on even-dimensional manifolds $\mathcal{M}^n$, such that $n = 2m$, $m \in \mathbb{N}$.

A function $N(\mathbf{x})$ such that

$$\{N, f\} \equiv 0$$

for any smooth function f on $\mathcal{M}^n$ is called ***annihilator*** or ***Casimir function*** of the Poisson bracket on $\mathcal{M}^n$. It is easy to see that every Casimir function should satisfy in local coordinates the equation

$$J^{ij} \frac{\partial N}{\partial x^j} \equiv 0.$$

A Poisson bracket has constant rank $l = 2s$ on $\mathcal{M}^n$ if $\operatorname{rank} ||J^{ij}|| = l$ everywhere on $\mathcal{M}^n$. In this case locally there always exist exactly $n - l$ independent functions $(N^1(\mathbf{x}), \dots, N^{n-l}(\mathbf{x}))$ which give a set of local Casimir functions for the corresponding chart of $\mathcal{M}^n$.

The common level surfaces of the local Casimir functions

$$N^1(\mathbf{x}) = \text{const}, \ \dots, \ N^{n-l}(\mathbf{x}) = \text{const}$$

represent an integrable foliation which is uniquely globally defined on the manifold $\mathcal{M}^n$. However, this does not mean necessarily that the functions $(N^1, \dots, N^{n-l})$ can be globally defined on the manifold $\mathcal{M}^n$, since the choice of independent set $(N^1(\mathbf{x}), \dots, N^{n-l}(\mathbf{x}))$ is individual for every local chart of $\mathcal{M}^n$. Thus, it is not possible to state in general that a Poisson bracket of constant rank l on $\mathcal{M}^n$ has $n - l$ globally defined Casimir functions on this manifold. This means certainly, that the corresponding foliation given by a Poisson structure of constant rank in general can not be defined as a set of common level surfaces of a set of global functions $(N^1, \dots, N^{n-l})$ on $\mathcal{M}^n$.

The important fact is that in general even the gradients of Casimir functions can not be globally defined on $\mathcal{M}^n$ as a set of global closed 1-forms on $\mathcal{M}^n$. As a result, the foliations defined by the Casimir functions of a Poisson structure on a manifold can in general be topologically more complicated than the foliations defined by a set of closed 1-forms on $\mathcal{M}^n$.

For non-degenerate Poisson structure on $\mathcal{M}^n$ the form

$$\omega_{ij} = ||J^{ij}||^{-1}$$

can be defined everywhere on $\mathcal{M}^n$. For any non-degenerate Poisson structure on $\mathcal{M}^n$ the form ω_{ij} is a globally defined non-degenerate closed 2-form on $\mathcal{M}^n$. The manifold $\mathcal{M}^n$ is called in this case a ***symplectic manifold*** and the form ω_{ij} gives the ***symplectic form*** of $\mathcal{M}^n$. Vice versa, on every symplectic manifold $\mathcal{M}^n$ the non-degenerate Poisson structure can be defined.

Every smooth function f on a Poisson manifold $\mathcal{M}^n$ generates a smooth vector field $\boldsymbol{\xi}_f$ according to the formula

$$\xi^i_f(\mathbf{x}) = J^{ij}(\mathbf{x})\frac{\partial f}{\partial x^j}.$$

The vector field $\boldsymbol{\xi}_f(\mathbf{x})$ is called the ***Hamiltonian vector field*** generated by f and the function f is called its ***Hamiltonian function***. The vector field $\boldsymbol{\xi}(\mathbf{x})$ is called ***locally Hamiltonian*** if in the vicinity of every point $\mathbf{x}_0 \in \mathcal{M}^n$ there exists a local function $f(\mathbf{x})$ which gives locally a Hamiltonian function for the field $\boldsymbol{\xi}(\mathbf{x})$.

Every locally Hamiltonian vector on $\mathcal{M}^n$ generates a one-parametric group of transformations of $\mathcal{M}^n$ which preserves the Poisson structure on $\mathcal{M}^n$. The last statement follows immediately from the fact that the Lie derivative of the tensor J^{ij} along the vector field $\boldsymbol{\xi}$ vanishes identically in this case.

The remarkable fact, which follows from the Jacobi identity for J^{ij} on $\mathcal{M}^n$, is that the mapping

$$f \to \boldsymbol{\xi}_f$$

defines the homomorphism of Lie algebras from the Lie algebra of functions to the Lie algebra of the vector fields on $\mathcal{M}^n$. So the vector field

$$[\boldsymbol{\xi}_f, \boldsymbol{\xi}_g]^i = \xi^j_f\frac{\partial \xi^i_g}{\partial x^j} - \xi^j_g\frac{\partial \xi^i_f}{\partial x^j}$$

is a Hamiltonian vector field with the Hamiltonian function $h = \{f, g\}$.

The analogous statement is true also for locally Hamiltonian vector fields. Thus, both Hamiltonian and locally Hamiltonian vector fields on $\mathcal{M}^n$ give the sub-algebras in the Lie algebra of the vector fields on $\mathcal{M}^n$.

The Poisson bracket of two functions $\{f, g\}$ has a very important meaning. Namely, it gives the derivative of the function f along the vector field $\boldsymbol{\xi}_g$ generated by g. In particular, the function f gives a conservation laws for the dynamical system corresponding to $\boldsymbol{\xi}_g$ if and only if $\{f, g\} \equiv 0$ everywhere on $\mathcal{M}^n$. It is easy to see then that the function g always gives the conservation law for $\boldsymbol{\xi}_g$ which represents the conservation of energy for the Hamiltonian vector field.

It is not difficult to show by use of Jacobi identity that the Poisson bracket of any two conservation laws for the vector field $\boldsymbol{\xi}(\mathbf{x})$ gives also a conservation law for the same vector field. So, the conservation laws for the Hamiltonian vector field $\boldsymbol{\xi}(\mathbf{x})$ generated by any function $g(\mathbf{x})$ on $\mathcal{M}^n$ represent always the Lie sub-algebra in the Lie algebra of functions on $\mathcal{M}^n$.

The canonical form of the non-degenerate Poisson bracket on a manifold $\mathcal{M}^n = \mathcal{M}^{2m}$ is given by the following theorem.

Theorem 1 (Darboux)**.** *For any non-degenerate Poisson bracket on the manifold $\mathcal{M}^{2m}$ there exist locally the coordinates $(q^1, \ldots, q^m, p_1, \ldots, p_m)$ such that*

$$\{q^\alpha, q^\beta\} = 0, \quad \{p_\alpha, p_\beta\} = 0, \quad \{q^\alpha, p_\beta\} = \delta^\alpha_\beta, \quad \alpha, \beta = 1, \ldots, m$$

where δ^α_β is the Kronecker symbol.

Darboux theorem can be generalized also to the case of constant rank Poisson brackets by use of the locally defined Casimir functions.

Theorem 2. *For any Poisson bracket of constant rank $l = 2s$ on a manifold $\mathcal{M}^n$ there exist local coordinates $(N^1, \ldots, N^{n-2s}, q^1, \ldots, q^s, p_1, \ldots, p_s)$ such that*

$$\{N^\lambda, N^\mu\} = 0, \quad \{N^\lambda, q^\alpha\} = 0, \quad \{N^\lambda, p_\alpha\} = 0,$$
$$\{q^\alpha, q^\beta\} = 0, \quad \{p_\alpha, p_\beta\} = 0, \quad \{q^\alpha, p_\beta\} = \delta^\alpha_\beta,$$

where $\lambda, \mu = 1, \ldots, N - 2s$ and $\alpha, \beta = 1, \ldots, s$.

It is easy to see that the coordinates $(N^1, \dots, N^{n-2s})$ play the role of local Casimir functions in this case. The Poisson bracket of the functions $(q^1, \dots, q^s, p_1, \dots, p_s)$ gives a non-degenerate Poisson bracket on every surface $N^1 = \text{const}, \dots, N^{n-2s} = \text{const}$. This bracket is called a restriction of the Poisson bracket of constant rank to the common level surfaces of Casimir functions.

The non-degenerate Poisson brackets in the canonical form are closely connected with the Lagrangian approach in classical mechanics. Let $\mathbf{q} = (q^1, \dots, q^m)$ be the generalized coordinates of a mechanical system and K and Π be, respectively, its kinetic and potential energy. Then the famous Lagrangian functional

$$S = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}})\, dt = \int_{t_1}^{t_2} (K(\mathbf{q}, \dot{\mathbf{q}}) - \Pi(\mathbf{q}))\, dt$$

leads to the non-degenerated Poisson structure through the Legendre transformation. Namely, as is well known, the Lagrangian equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, s$$

are equivalent to the equations

$$\dot{q}^i = \frac{\partial H(\mathbf{q}, \mathbf{p})}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H(\mathbf{q}, \mathbf{p})}{\partial q_i}$$

after the transformation

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad H(\mathbf{q}, \mathbf{p}) \equiv L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p})) - \dot{q}^i(\mathbf{q}, \mathbf{p})\frac{\partial L}{\partial \dot{q}^i}(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}))$$

The transformation from the Lagrangian formalism to non-degenerate Hamiltonian formalism can be made also in more general case of Lagrangian functions depending on higher derivatives of the coordinates $\mathbf{q}$ by use of Ostrogradskii transformations.

Example 3. ***Lie–Poisson bracket.*** The most important examples of the Poisson brackets of constant rank are given by the Poisson brackets defined by the Lie algebras. Namely, if $\mathcal{L}$ is a Lie algebra with a basic $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ and the structure constants C^i_{jk}

$$[\mathbf{e}_j, \mathbf{e}_k] = C^i_{jk}\mathbf{e}_i$$

(summation over repeated indices), then the natural Poisson bracket on the dual space $\mathcal{L}^*$ with coordinates $(x_1, \ldots, x_n)$ can be defined as

$$\{x_j, x_k\} = C^i_{jk} x_i.$$

The Casimir functions of the universal enveloping algebra play then the role of the natural annihilators of this Poisson bracket.

The Hamiltonian formulation plays an important role in the definition of complete integrability of dynamical system, see Liouville integrability. In particular, the nice construction underlying the integrability in many important examples is the bi-Hamiltonian structure.

2. Discrete infinite-dimensional Poisson brackets

The infinite-dimensional Poisson brackets are the generalizations of the finite-dimensional Poisson brackets where the number of coordinates is infinite. The same properties of the skew-symmetry and Jacobi identity are also required in the infinite-dimensional case. As a rule, the infinite-dimensional Poisson brackets arise as the field-theoretical (discrete or continuous) Poisson brackets for the field-theoretical systems or PDE's. As an example, let us consider the infinite-dimensional discrete Poisson bracket on the space of fields $\boldsymbol{\varphi}(k) = (\varphi^1(k), \ldots, \varphi^n(k))$ where k is the integer number $k \in \mathbb{Z}$ numerating the positions of cells in the case of one spatial dimension.

The values $\boldsymbol{\varphi}(k)$ play now the role of "coordinates" in the functional space and the general form of the Poisson bracket can be written as

$$\{\varphi^i(k), \varphi^j(l)\} = J^{ij}_{kl}[\boldsymbol{\varphi}] \tag{3}$$

where $J^{ij}_{kl}[\boldsymbol{\varphi}]$ are some functionals on the functional space $\{\boldsymbol{\varphi}(k)\}$.

In most of important cases all functionals $J^{ij}_{kl}[\boldsymbol{\varphi}]$ depend just on the finite number of field variables $\boldsymbol{\varphi}(k')$ such that the Jacobi identity has a normal form (2).

Bracket (3) is called local if

$$J^{ij}_{kl} \equiv 0, \quad |k - l| > N_1, \quad \frac{\partial J^{ij}_{kl}}{\partial \varphi^q(m)} \equiv 0, \quad |k - m| > N_2$$

for some N_1, N_2. Bracket (3) is called translational invariant if it is invariant under all (integral) shifts of the field index k: $\varphi^i(k) \to \varphi^i(k+k_0)$.

The general form of a dynamical system generated by a functional $H = H[\boldsymbol{\varphi}]$ can be written in a natural way

$$\dot{\varphi}^i(k) = \sum_{j,l} J^{ij}_{kl}[\boldsymbol{\varphi}] \frac{\partial H}{\partial \varphi^j(l)}.$$

The functional $H[\boldsymbol{\varphi}]$ is usually called local functional if it is of the form

$$H[\boldsymbol{\varphi}] = \sum_{k=-\infty}^{\infty} h_k[\boldsymbol{\varphi}]$$

where

$$\frac{\partial h_k}{\partial \varphi^i(l)} \equiv 0, \quad |k-l| > N$$

for some $N > 0$. In the same way, the functional $H[\boldsymbol{\varphi}]$ is called translational invariant functional if it is invariant under the integral shifts $\varphi^i(k) \to \varphi^i(k+k_0)$.

The local translational invariant brackets (3) play very important role in the theory of one-dimensional discrete dynamical systems both in integrable and non-integrable cases. It is easy to see also how the definitions above can be generalized to the case of several spatial variables.

3. Local field-theoretical Poisson brackets and symplectic structures

The continuous version of the infinite-dimensional Poisson brackets can be defined on the space of smooth functions $\boldsymbol{\varphi}(x) = (\varphi^1(k), \ldots, \varphi^n(x))$, $x \in \mathbb{R}$. The values $\boldsymbol{\varphi}(x)$ can be considered then as the "coordinates" on this functional space and the general form of the infinite-dimensional Poisson bracket can be written as

$$\{\varphi^i(x), \varphi^j(y)\} = J^{ij}[\boldsymbol{\varphi}](x,y) \tag{4}$$

where $J^{ij}(x,y)$ are some distributions on $\mathbb{R} \times \mathbb{R}$.

The skew-symmetry properties and the Jacobi identity

$$\{\varphi^i(x), \{\varphi^j(y), \varphi^k(z)\}\} + \{\varphi^j(y), \{\varphi^k(z), \varphi^i(x)\}\} + \{\varphi^k(z), \{\varphi^i(x), \varphi^j(y)\}\} = 0$$

are required in the sense of distributions in this case.

As a rule, brackets (4) are considered on the space of rapidly decreasing or periodic functions $\boldsymbol{\varphi}(x)$.

The functional $I[\boldsymbol{\varphi}]$ is called here the smooth functional if the variational derivatives $\delta I/\delta\varphi^i(x)$ are the smooth functions of x. The Poisson bracket of the smooth functionals can be formally written as

$$\{I, J\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\delta I}{\delta\varphi^i(x)} J^{ij}[\boldsymbol{\varphi}](x,y) \frac{\delta J}{\delta\varphi^j(y)}\, dx\, dy \equiv \mathbf{J}^{ij}[\boldsymbol{\varphi}]\left(\frac{\delta I}{\delta\varphi^i(x)} \otimes \frac{\delta J}{\delta\varphi^j(y)}\right)$$

where $\mathbf{J}^{ij}$ is a functional corresponding to the distribution $J^{ij}(x,y)$.

Poisson bracket (4) is called local if it is of the form

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k\geq 0} B^{ij}_{(k)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)\delta^{(k)}(x-y) \tag{5}$$

where $B^{ij}_{(k)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)$ are some smooth functions of $(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)$ and $\delta^{(k)}(x-y) \equiv \partial^k/\partial x^k \delta(x-y)$. It is assumed also that all $B^{ij}_{(k)}$ depend on the finite number of arguments and the sum contains just the finite number of terms.

The corresponding Hamiltonian operator $\hat{J}^{ij}$ can be written in this case as

$$\hat{J}^{ij} = \sum_{k\geq 0} B^{ij}_{(k)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)\frac{\partial^k}{\partial x^k}$$

and the Poisson bracket of two smooth functionals I, J can be written then in the form

$$\{I, J\} = \int_{-\infty}^{+\infty} \frac{\delta I}{\delta\varphi^i(x)} \sum_{k\geq 0} B^{ij}_{(k)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)\frac{\partial^k}{\partial x^k}\frac{\delta J}{\delta\varphi^j(x)}\, dx$$

The general form of dynamical system generated by the "local" functional H

$$H = \int_{-\infty}^{+\infty} h(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)\, dx$$

can be represented as

$$\dot{\varphi}^i = \sum_{k \geq 0} B^{ij}_{(k)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots) \frac{\partial^k}{\partial x^k} \frac{\delta H}{\delta \varphi^j(x)}$$

and gives a local expression for the "vector field" $\boldsymbol{\xi}^i(x)$ generated by H.

Brackets (5) play an important role both for integrable and non-integrable evolution systems, however, the complete theory of these brackets is still absent at the moment.

Another important object is the local symplectic form on the space of fields $\varphi^i(x)$ having the form

$$\Omega_{ij}(x, y) = \sum_{k \geq 0} \omega^{(k)}_{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)\, \delta^{(k)}(x - y). \tag{6}$$

The corresponding symplectic operator can be written as

$$\hat{\Omega}_{ij} = \sum_{k \geq 0} \omega^{(k)}_{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots) \frac{\partial^k}{\partial x^k}$$

and the connection between the time-derivative of $\varphi^i(x)$ and Hamiltonian functional H is given by

$$\sum_{k \geq 0} \omega^{(k)}_{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots) \frac{\partial^k}{\partial x^k} \dot{\varphi}^j = \frac{\delta H}{\delta \varphi^i(x)}.$$

Symplectic form (6) should satisfy the skew-symmetry and closeness conditions on the space of functions $\varphi^i(x)$:

$$\Omega_{ij}(x, y) = -\Omega_{ji}(y, x), \qquad \frac{\delta \Omega_{ij}(x, y)}{\delta \varphi^k(z)} + \frac{\delta \Omega_{jk}(y, z)}{\delta \varphi^i(x)} + \frac{\delta \Omega_{ki}(z, x)}{\delta \varphi^j(y)} = 0.$$

4. Poisson brackets of hydrodynamic type

An important class of local field-theoretical Poisson brackets is given by the local Poisson brackets of ***hydrodynamic type***, or ***Dubrovin–Novikov*** brackets [6, 7].

Definition 4. The Dubrovin–Novikov bracket (DN-bracket) is a bracket on the functional space $(U^1(x), \ldots, U^N($ of the form

$$\{U^\nu(x), U^\mu(y)\} = g^{\nu\mu}(\mathbf{U})\delta'(x-y) + b^{\nu\mu}_\lambda(\mathbf{U})U^\lambda_x\delta(x-y). \tag{7}$$

The bracket (7) is called non-degenerate if $\det ||g^{\nu\mu}(\mathbf{U})|| \neq 0$.

The corresponding Hamiltonian operator $\hat{J}^{\nu\mu}$ can be written as

$$\hat{J}^{\nu\mu} = g^{\nu\mu}(\mathbf{U})\frac{\partial}{\partial x} + b^{\nu\mu}_\lambda(\mathbf{U})U^\lambda_x$$

and is homogeneous w.r.t. transformation $x \to ax$.

Every functional H of hydrodynamic type, that is the functional of the form

$$H = \int_{-\infty}^{+\infty} h(\mathbf{U})dx$$

generates a system of hydrodynamic type according to the formula

$$U^\nu_t = \hat{J}^{\nu\mu}\frac{\delta H}{\delta U^\mu(x)} = g^{\nu\mu}(\mathbf{U})\frac{\partial}{\partial x}\frac{\partial h}{\partial U^\mu} + b^{\nu\mu}_\lambda(\mathbf{U})\frac{\partial h}{\partial U^\mu}U^\lambda_x. \tag{8}$$

It was shown by Dubrovin and Novikov themselves that the theory of DN-brackets is closely connected with Riemannian geometry. In fact, it follows from the skew-symmetry of (7) that the coefficients $g^{\nu\mu}(\mathbf{U})$ give in the non-degenerate case the contravariant pseudo-Riemannian metric on the manifold $\mathcal{M}^N$ with coordinates $(U^1, \ldots, U^N)$ while the functions

$$\Gamma^\nu_{\mu\lambda}(\mathbf{U}) = -g_{\mu\alpha}(\mathbf{U})b^{\alpha\nu}_\lambda(\mathbf{U})$$

(where $g_{\nu\mu}(\mathbf{U})$ is the corresponding metric with lower indices) give the connection coefficients compatible with metric $g_{\nu\mu}(\mathbf{U})$. The validity of Jacobi identity requires then that $g_{\nu\mu}(\mathbf{U})$ is actually a flat metric on the manifold $\mathcal{M}^N$ and the functions $\Gamma^{\nu}_{\mu\lambda}(\mathbf{U})$ give a symmetric (Levi–Civita) connection on $\mathcal{M}^N$.

In the flat coordinates $n^1(\mathbf{U}),\ldots,n^N(\mathbf{U})$ the non-degenerate DN-bracket can be written in constant form:

$$\{n^{\nu}(x), n^{\mu}(y)\} = e^{\nu}\delta^{\nu\mu}\delta'(x-y), \quad e^{\nu} = \pm 1.$$

The functionals

$$N^{\nu} = \int_{-\infty}^{+\infty} n^{\nu}(x)\, dx$$

are the annihilators of the bracket (7) and the functional

$$P = \frac{1}{2}\int_{-\infty}^{+\infty}\sum_{\nu=1}^{N} e^{\nu}(n^{\nu}(x))^2\, dx$$

is the momentum functional generating the system $U^{\nu}_t = U^{\nu}_x$ according to (8).

Another important choice of coordinates for DN-bracket is given by the so-called “physical” or “Liouville” coordinates. This type of coordinates is usually associated with the densities of conservation laws of the hydrodynamic systems. We say that the coordinates are “Liouville” or “physical” for the DN-bracket if the bracket has the form:

$$\{U^{\nu}(X), U^{\mu}(Y)\} = (\gamma^{\nu\mu}(U) + \gamma^{\mu\nu}(U))\delta'(X-Y) + \frac{\partial\gamma^{\nu\mu}}{\partial U^{\lambda}} U^{\lambda}_X \delta(X-Y)$$

for some functions $\gamma^{\nu\mu}(U)$. Any coordinates such that integrals of them define the commuting flows, are physical in that sense.

Let us mention also that the degenerate brackets (7) are more complicated but have a nice differential geometric structure as well.

The brackets (7) are closely connected with the integration theory of systems of hydrodynamic type

$$U^{\nu}_t = V^{\nu}_{\mu}(\mathbf{U})U^{\mu}_x. \tag{9}$$

Namely, according to conjecture of S.P. Novikov, all diagonalizable systems (9) which are Hamiltonian with respect to DN-brackets (7) (with Hamiltonian function of hydrodynamic type) are completely integrable. This conjecture was proved by S.P. Tsarev who proposed a general procedure called ***generalized hodograph method*** of integration of Hamiltonian diagonalizable systems (9).

In fact the generalized hodograph method permits to integrate the wider class of diagonalizable systems (9) (semi-Hamiltonian systems) which appeared to be Hamiltonian in more general (weakly nonlocal) Hamiltonian formalism.

The symplectic structure corresponding to non-degenerate DN-bracket has the "weakly nonlocal" form and can be written in coordinates n^ν as

$$\Omega_{\nu\mu}(x,y) = e^\nu \delta_{\nu\mu}\sigma(x-y), \quad \sigma(x-y) = 1/2\,\mathrm{sign}(x-y).$$

More generally, in arbitrary coordinates U^ν one has

$$\Omega_{\nu\mu}(x,y) = \sum_{\lambda=1}^{N} e^\lambda \frac{\partial n^\lambda}{\partial U^\nu}(x)\sigma(x-y)\frac{\partial n^\lambda}{\partial U^\mu}(y).$$

5. Weakly nonlocal Poisson brackets and symplectic structures

The field-theoretical Poisson bracket is called ***weakly nonlocal*** [8] if it can be written in the form

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k\ge 0} B_k^{ij}(\varphi, \varphi_x, \dots)\delta^{(k)}(x-y) + \sum_{k\ge 0} e_k S^i_{(k)}(\varphi, \varphi_x, \dots)\sigma(x-y)S^j_{(k)}(\varphi, \varphi_y, \dots) \qquad (10)$$

where $e_k = \pm 1$, $\sigma(x-y) = -\sigma(y-x)$, $\partial_x\sigma(x-y) = \delta(x-y)$ and both sums contain the finite numbers of terms depending on the finite numbers of derivatives of φ with respect to x. It is assumed also that the "vector-fields"

$$\mathbf{S}_{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots) = \left(S^1_{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots), \dots S^n_{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)\right)^t$$

are linearly independent (over constant coefficients).

We can introduce also the Hamiltonian operator $\hat{J}^{ij}$:

$$\hat{J}^{ij} = \sum_{k\geq 0} B^{ij}_{(k)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)\frac{\partial^k}{\partial x^k} + \sum_{s=1}^{g} e_s S^i_{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots) D^{-1} S^j_{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots) \tag{11}$$

where D^{-1} is the integration operator defined in the skew-symmetric way:

$$D^{-1}\xi(x) = \frac{1}{2}\int_{-\infty}^{x} \xi(y)dy - \frac{1}{2}\int_{x}^{+\infty} \xi(y)dy.$$

For the functional $H[\varphi]$ the corresponding dynamical system can be written in the form:

$$\begin{aligned}\varphi^i_t = \hat{J}^{ij}\frac{\delta H}{\delta\varphi^j(x)} &= \sum_{k\geq 0} B^{ij}_{(k)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)\frac{\partial^k}{\partial x^k}\frac{\delta H}{\delta\varphi^j(x)} \\ &+ \sum_{s=1}^{g} e_s S^i_{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots) D^{-1}\Big[S^j_{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)\frac{\delta H}{\delta\varphi^j(x)}\Big].\end{aligned} \tag{12}$$

As previously, operator (11) should also be skew-symmetric and satisfy the Jacobi identity.

It is not difficult to see that the functional

$$H = \int_{-\infty}^{+\infty} h(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)dx \tag{13}$$

generates a local dynamical system

$$\varphi^i_t = S^i(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)$$

according to (12) if it gives a conservation law for all the dynamical systems

$$\varphi^i_{t_s} = S^i_{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)$$

that is

$$h_{t_s} \equiv \partial_x Q_s(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)$$

for some functions $Q_s(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)$.

Let us formulate the result which relates the non-local and local parts of the general weakly-nonlocal Poisson brackets (10).

Theorem 5 ([9]). *For any bracket (10) the flows*

$$\varphi^i_{t_s} = S^i_{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots) \tag{14}$$

commute with each other and leave the bracket (10) invariant.

It can be easily proved also that the flows (14) commute with any local Hamiltonian flow in the structure (10) generated by local Hamiltonian functional (13).

The general weakly nonlocal Poisson bracket of hydrodynamic type (Ferapontov bracket) has the form ($e_k = \pm 1$)

$$\{U^\nu(x), U^\mu(y)\} = g^{\nu\mu}(U)\delta'(x-y) + b^{\nu\mu}_\lambda(U)U^\lambda_x\delta(x-y) + \sum_{k=1}^{g} e_k w^\nu_{(k)\lambda}(U)U^\lambda_x\sigma(x-y)w^\mu_{(k)\delta}(U)U^\delta_y. \tag{15}$$

The statements analogous to the local situation can be proved also in the case of the bracket (15).

Theorem 6 ([10]). *The bracket (15) satisfies the Jacobi Identity if and only if the values $\Gamma^\nu_{\mu\lambda} = -g_{\mu\alpha}b^{\alpha\nu}_\lambda$ give the Cristofel connection for the metric $g^{\nu\mu}$ and the metric $g^{\nu\mu}$ (with lower indices) and tensors $w^\nu_{(k)\mu}$ satisfy the equations:*

$$g_{\nu\tau}w^\tau_{(k)\mu} = g_{\mu\tau}w^\tau_{(k)\nu}, \qquad \nabla_\nu w^\mu_{(k)\lambda} = \nabla_\lambda w^\mu_{(k)\nu}, \qquad R^{\nu\tau}_{\mu\lambda} = \sum_{k=1}^{g} e_k\Big(w^\nu_{(k)\mu}w^\tau_{(k)\lambda} - w^\tau_{(k)\mu}w^\nu_{(k)\lambda}\Big).$$

Moreover, this set is commutative, $[w_k, w_{k'}] = 0$.

The equations written above are the Gauss–Codazzi equations for the submanifolds $\mathcal{M}^N$ with flat normal connection in the Pseudo-Euclidean space E^{N+g}. Here $g_{\nu\mu}$ is the first quadratic form of $\mathcal{M}^N$, and $w_{(k)}$ are the Weingarten operators corresponding to the field of pairwise orthogonal unit normals $\vec{n}_k$. Moreover, it was proved by E.V. Ferapontov that these brackets can be obtained as a result of the Dirac restriction of the local DN-bracket

$$\{N^I(x), N^J(y)\} = \epsilon^I \delta^{IJ} \delta'(x-y), \quad I, J = 1, \dots, N+g, \quad \epsilon^I = \pm 1$$

in E^{N+g} to the corresponding submanifold $\mathcal{M}^N$.

The canonical form of bracket (15) (F-bracket) can be written in the form analogous to the canonical form of DN-bracket. However, some new special features arise in this situation.

Definition 7. We say that the F-bracket is written in the Canonical form if

$$\{n^\nu(x), n^\mu(y)\} = \left(\epsilon^\nu \delta^{\nu\mu} - \sum_{k=0}^{g} e_k f^\nu_{(k)}(n) f^\mu_{(k)}(n)\right) \delta'(x-y)$$
$$- \sum_{k=0}^{g} e_k \big(f^\nu_{(k)}(n)\big)_x f^\mu_{(k)}(n) \delta(x-y) + \sum_{k=0}^{g} e_k \big(f^\nu_{(k)}(n)\big)_x \sigma(x-y) \big(f^\mu_{(k)}(n)\big)_y \tag{16}$$

with non-degenerate metric and some functions $f^\nu_{(k)}(n)$ such that $f^\nu_{(k)}(0) \equiv 0$, $e_k = \pm 1$.

The following theorem can be proved about the canonical form of the F-bracket.

Theorem 8 ([8]). *I) Every F-bracket (15) with the non-degenerate metric tensor $g^{\nu\mu}(U)$ can be locally written in the canonical form (16) after some coordinate transformation $n^\nu = n^\nu(U)$. Moreover, for any given point U_0 it is possible to choose the coordinates $n^\nu(U)$ in such a way that $n^\nu(U_0) \equiv 0$, $f^\nu_{(k)}(U_0) \equiv 0$.*

II) The integrals

$$N^\nu = \int n^\nu(X) dX$$

are annihilators of bracket (16) on the domain in the space of rapidly decreasing functions $n^\nu(X)$ bounded by the small enough constant;

III) The flows

$$n^\nu_{t_k} = \frac{d}{dX} f^\nu_{(k)}(n)$$

are generated by the local Hamiltonians

$$H_k = \int h_k(n)dX$$

on the same phase space. The functions $n^\nu(U)$, $h_k(n(U))$ can be represented as linear combinations of coordinates V^I in the pseudo-Euclidean space $\mathbb{E}^{N+g}$ for the local representation of our manifold as a submanifold $M^N \subset \mathbb{E}^{N+g}$ with flat normal connection.

Geometrically, for any point $U_0 \in \mathcal{M}^N$ the flat coordinates of pseudo-Eucledian space $\mathbb{E}^{N+g}$ tangential to $\mathcal{M}^N$ at the point U_0 give the annihilators of the bracket (15) on the loop space $\{\gamma(x) \subset \mathcal{M}^N : \gamma(-\infty) = \gamma(+\infty) = U_0\}$, while the flat coordinates in $\mathbb{E}^{N+g}$ orthogonal to $\mathcal{M}^N$ at the point U_0 give the Hamiltonian functionals for the flows in the nonlocal tail of (15) on the same phase space.

The "physical" or "Liouville" coordinates for the weakly nonlocal Poisson brackets of hydrodynamic type are defined by the requirements that

$$\{U^\nu(X), U^\mu(Y)\} = \Big(\gamma^{\nu\mu}(X) + \gamma^{\mu\nu}(X) - \sum_{k=1}^{g} e_k f^\nu_{(k)} f^\mu_{(k)}\Big)\delta'(X-Y)$$
$$+\Big(\frac{\partial\gamma^{\nu\mu}}{\partial U^\lambda}U^\lambda_X - \sum_{k=1}^{g} e_k (f^\nu_{(k)})_X f^\mu_{(k)}\Big)\delta(X-Y) + \sum_{k=1}^{g} e_k (f^\nu_{(k)})_X \sigma(X-Y)(f^\mu_{(k)})_Y$$

for some functions $\gamma^{\nu\mu}(U)$ and $f^\nu_{(k)}(U)$.

Like in the local case, the bracket (15) of F-type has Physical form in the coordinates U^μ if and only if the integrals $J^\nu = \int U^\nu(X)dX$ generate the set of local commuting flows according to bracket (15).

Poisson brackets (15) are connected with the integrable systems of hydrodynamic type in the same way as the local Dubrovin–Novikov brackets. Namely, the Tsarev integration procedure based on the Riemannian metric turns out to be valid also for the case of weakly non-local Poisson brackets of hydrodynamic type. In fact, probably all semihamiltonian systems are Hamiltonian corresponding to some weakly nonlocal Poisson bracket of hydrodynamic type with (maybe) an infinite number of terms in the nonlocal tail.

One of the most famous weakly non-local Poisson brackets of hydrodynamic type is the Mokhov–Ferapontov bracket (MF-bracket) having the form

$$\{U^\nu(X), U^\mu(Y)\} = g^{\nu\mu}(U)\delta'(X-Y) + b^{\nu\mu}_\lambda(U)U^\lambda_X\delta(X-Y) + cU^\nu_X\sigma(X-Y)U^\mu_Y. \tag{17}$$

For bracket (17) with non-degenerate metric tensor $g^{\nu\mu}(U)$ the following statements are true.

Theorem 9 ([11]). *Bracket (17) is skew-symmetric if and only if the tensor $g^{\nu\mu}$ is symmetric, and the connection*

$$\Gamma^{\nu}_{\mu\lambda} = -g_{\mu\tau} b^{\tau\nu}_{\lambda}$$

is compatible with this metric: $\nabla_\lambda g_{\mu\nu} \equiv 0$.

Bracket (17) satisfies the Jacobi identity if and only if its connection $\Gamma^{\nu}_{\mu\lambda}$ is symmetric (that is the torsion tensor vanishes) and has the constant curvature equal to c, that is

$$R^{\nu\tau}_{\mu\lambda} = c\left(\delta^{\nu}_{\mu}\delta^{\tau}_{\lambda} - \delta^{\tau}_{\mu}\delta^{\nu}_{\lambda}\right).$$

The canonical form of MF-bracket was first pointed out by M.V. Pavlov and can be written as

$$\{n^\nu(X), n^\mu(Y)\} = \left(\epsilon^\nu \delta^{\nu\mu} - c n^\nu n^\mu\right)\delta'(X-Y) - c n^\nu_X n^\mu \delta(X-Y) + c n^\nu_X \sigma(X-Y) n^\mu_Y \tag{18}$$

where ϵ^ν are equal to ± 1, and the term $\epsilon^\nu \delta^{\nu\mu}$ has the same signature as metric tensor $g^{\nu\mu}(U)$.

The functionals

$$N^\nu = \int n^\nu dX$$

are the annihilators of the bracket (18). The functional

$$P = \frac{1}{c}\int\left(1 - \sqrt{\left|1 - c\sum_{\nu=1}^{N} \epsilon^\nu n^\nu(X) n^\nu(X)\right|}\right) dX$$

is the momentum generating shifts along the coordinate X [12].

Geometrically, the MF-bracket corresponds to a restriction of a DN-bracket to a (pseudo-)sphere $\mathbb{S}^N \in \mathbb{E}^{N+1}$ of codimension 1 in the (pseudo-)Euclidean space.

The general weakly-nonlocal symplectic structures have the form

$$\Omega_{ij}(x,y) = \sum_{k\geq 0} \omega^{(k)}_{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)\delta^{(k)}(x-y) + \sum_{s=1}^{g} e_s q^{(s)}_i(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)\sigma(x-y) q^{(s)}_j(\boldsymbol{\varphi}, \boldsymbol{\varphi}_y, \dots) \tag{19}$$

where $\boldsymbol{\varphi} = (\varphi^1, \dots, \varphi^n)$, $i, j = 1, \dots, n$, $e_s = \pm 1$, $\sigma(x - y) = 1/2 \operatorname{sign}(x - y)$ and $\omega_{ij}^{(k)}$ and $q_i^{(s)}$ are some local functions of $\boldsymbol{\varphi}$ and it's derivatives at the same point. It is assumed also that both sums contain finite number of terms and all $\omega_{ij}^{(k)}$ and $q_i^{(s)}$ depend on finite number of derivatives of $\boldsymbol{\varphi}$. We also assume here that the non-local part of (19) is written in the "irreducible" form such that the 1-forms $\{\mathbf{q}^{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)\}$ give a linearly independent set (with constant coefficients).

The following general theorem can be formulated in this case.

Theorem 10 ([13]). *For any closed 2-form (19) the functions $q_i^{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)$ represent the closed 1-forms, that is*

$$\frac{\delta q_i^{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)}{\delta \varphi^j(y)} - \frac{\delta q_j^{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_y, \dots)}{\delta \varphi^i(x)} \equiv 0.$$

The weakly nonlocal symplectic structures of hydrodynamic type have the form:

$$\Omega_{\nu\mu}(X, Y) = \sum_{s,p=1}^{M} \kappa_{sp} \omega_\nu^{(s)}(\mathbf{U}(X)) \sigma(X - Y) \omega_\mu^{(p)}(\mathbf{U}(Y))$$

or in "diagonal" form

$$\Omega_{\nu\mu}(X, Y) = \sum_{s=1}^{M} e_s \omega_\nu^{(s)}(\mathbf{U}(X)) \sigma(X - Y) \omega_\mu^{(s)}(\mathbf{U}(Y)) \tag{20}$$

in coordinates U^ν where κ_{sp} is some quadratic form, $e_s = \pm 1$, and $\omega_\nu^{(s)}(\mathbf{U})$ are closed 1-forms on the manifold $\mathcal{M}^N$. Locally the forms $\omega_\nu^{(s)}(\mathbf{U})$ can be represented as the gradients of some functions $f^{(s)}(\mathbf{U})$ such that

$$\Omega_{\nu\mu}(X, Y) = \sum_{s=1}^{M} e_s \frac{\partial f^{(s)}}{\partial U^\nu}(X) \sigma(X - Y) \frac{\partial f^{(s)}}{\partial U^\mu}(Y). \tag{21}$$

The following general theorem can be formulated for the weakly non-local symplectic structures of hydrodynamic type.

Theorem 11 ([14, 13]). *Expression (20) gives the closed 2-form on the space $\{\mathbf{U}(X)\}$ if and only if the 1-forms $\omega_\nu^{(s)}(\mathbf{U})$ on $\mathcal{M}^N$ are closed,*[1] *that is*

$$\frac{\partial}{\partial U^\nu}\omega_\mu^{(s)}(\mathbf{U}) = \frac{\partial}{\partial U^\mu}\omega_\nu^{(s)}(\mathbf{U}).$$

The 2-form $\Omega_{\nu\mu}(X,Y)$ written in form (21) can be considered as the pullback of the form

$$\Xi_{IJ}(X,Y) = e_I\delta_{IJ}\sigma(X-Y), \quad I,J = 1,\dots,M$$

defined in the pseudo-Euclidean space $\mathbb{E}^N$ with the metric $G_{IJ} = \operatorname{diag}(e_1,\dots,e_M)$ for the mapping $\alpha : \mathcal{M}^N \to \mathbb{E}^N$

$$(U^1,\dots,U^N) \to (f^{(1)}(\mathbf{U}),\dots,f^{(M)}(\mathbf{U})).$$

Definition 12. We call symplectic form (20) non-degenerate if $M \geq N$ and

$$\operatorname{rank}\begin{pmatrix}\omega_i^{(1)}(\mathbf{U})\\ \dots\\ \omega_i^{(M)}(\mathbf{U})\end{pmatrix} = N.$$

The non-degenerate symplectic forms (20) are closely connected with the weakly nonlocal Poisson brackets of hydrodynamic type (15). Namely, as can be shown, the symplectic form for the bracket (15) can be written in the form

$$\Omega_{\nu\mu}(X,Y) = \sum_{I=1}^{N+g} e^I\frac{\partial V^I}{\partial U^\nu}(X)\sigma(X-Y)\frac{\partial V^I}{\partial U^\mu}(Y)$$

$$= \sum_{\tau=1}^{N} e^\tau\frac{\partial n^\tau}{\partial U^\nu}(X)\sigma(X-Y)\frac{\partial n^\tau}{\partial U^\mu}(Y) + \sum_{k=1}^{g} e_k\frac{\partial h_k}{\partial U^\nu}(X)\sigma(X-Y)\frac{\partial h_k}{\partial U^\mu}(Y)$$

where V^I are the coordinates in the pseudo-Euclidean space $\mathbb{E}^{N+g}$ for the local representation of our manifold as a submanifold $M^N \subset \mathbb{E}^{N+g}$ with flat normal connection [8].

[1] We assume that (20) is written in the "irreducible" form, i.e. the 1-forms $\omega_\nu^{(s)}(\mathbf{U})$ are linearly independent (with constant coefficients).

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77 Hénon–Heiles system

$$u'' = -au - 2duv, \quad v'' = -bv + cv^2 - du^2$$

- Hamiltonian: $H = \frac{1}{2}((u')^2 + (v')^2 + au^2 + bv^2) + du^2v - \frac{1}{3}cv^3$.
- The integrable cases: $d = -c$, $b = a$; $6d = -c$; $16d = -c$, $b = 16a$.

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78 Hirota equation

$$\alpha u_1 u_{23} + \beta u_2 u_{13} + \gamma u_3 u_{12} = 0, \quad u_i = T_i(u) \tag{1}$$

➢ The parameters may be dropped out by the scaling $u(i,j,k) \to u(i,j,k)\exp(\lambda ij + \mu ik + \nu jk)$.

➢ Equation (1) passes singularity confinement test [2] and possesses 4D-consistency property.

➢ The auxiliary linear problems [3, 4]:

$$\phi_1 = a\phi + \phi_2, \quad \phi_3 = b\phi + \phi_2 \quad \Rightarrow \quad a_3 + b_2 = a_2 + b_1, \quad a_3 b = ab_1.$$

➢ Considering the equations on a, b as the conservation laws suggests the substitution

$$a = \frac{u_{12}u}{u_1 u_2}, \quad b = \frac{u_{23}u}{u_2 u_3}$$

which brings to (1). Conversely, eliminating a and b brings to equation

$$\frac{\phi_{13} - \phi_{12}}{\phi_1} + \frac{\phi_{12} - \phi_{23}}{\phi_2} + \frac{\phi_{23} - \phi_{13}}{\phi_3} = 0. \tag{2}$$

Equation (1) can be considered as the limiting case of Hirota–Miwa equation (80.3) while (2) corresponds to the double cross-ratio equation (80.1).

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79 Hirota operator

The substitution $u = -2D_x^2(\log\tau)$ brings the KdV equation $u_t = u_{xxx} - 6uu_x$ to the ***bilinear form***

$$\tau\tau_{xt} - \tau_x\tau_t = \tau\tau_{xxxx} - 4\tau_x\tau_{xxx} + 3\tau_{xx}^2.$$

It can be conveniently written as

$$(D_xD_t - D_x^4)\tau\cdot\tau = 0$$

by use of ***Hirota operator*** which acts on the ordered product of two functions accordingly to the rule

$$D_x f\cdot g = f_x g - f g_x.$$

In a sense, this substitution is similar to the formula $\wp = -(\log\sigma)''$ in the theory of elliptic functions which represents Weierstrass $\wp$-function in terms of entire σ-function. Namely, it turns out that N-soliton solutions of KdV equation correspond just to the linear combination of exponentials

$$\tau = \sum_{\mu_j\in\{0,1\}} \exp\left(\sum_{j=1}^{N}\mu_j\theta_j + \sum_{1\le i<j\le N}\mu_i\mu_j A_{ij}\right)$$

where $\theta_j = k_j x + k_j^3 t + \delta_j$ are phases of solitons and $A_{ij} = 2\log\frac{k_i-k_j}{k_i+k_j}$ are phase shifts.

The analogous bilinear forms exist for many other integrable equations.

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80 Hirota–Miwa equation

Double cross-ratio equation
$$\frac{(\psi_i-\psi_k)(\psi_j-\psi_{ijk})}{(\psi_k-\psi_j)(\psi_{ijk}-\psi_i)}=\frac{(\psi-\psi_{ik})(\psi_{ij}-\psi_{jk})}{(\psi_{ik}-\psi_{ij})(\psi_{jk}-\psi)} \tag{1}$$

Star-triangle mapping
$$a^{ij}_k=-\frac{a^{ij}}{a^{ij}a^{jk}+a^{ki}a^{ij}+a^{jk}a^{ki}},\quad a^{ij}=-a^{ji} \tag{2}$$

Hirota–Miwa equation
$$uu_{ijk}=\varepsilon^{ij}\varepsilon^{ik}u_iu_{jk}+\varepsilon^{ji}\varepsilon^{jk}u_ju_{ik}+\varepsilon^{ki}\varepsilon^{kj}u_ku_{ij},\quad \varepsilon^{ij}=\mathrm{sign}(i-j) \tag{3}$$

It is assumed in all equations that $i\neq j\neq k\neq i$.

➢ The linear problem:
$$(T_iT_j+a^{ij}(T_i-T_j)-1)\psi=0. \tag{4}$$

The consistency condition $T_k(\psi_{ij})=T_j(\psi_{ik})$ leads to the star-triangle mapping. On the other hand, (4) allows to eliminate the variables a^{ij} and this leads to the double cross-ratio equation. The variable u is introduced due to the conservation laws
$$\frac{T_i(a^{jk})}{a^{jk}}=\frac{T_j(a^{ik})}{a^{ik}}=\frac{T_k(a^{ij})}{a^{ij}}\quad\Rightarrow\quad a^{ij}=\varepsilon^{ij}\frac{u_iu_j}{uu_{ij}},$$
resulting in the equation (3).

➢ Equations (1)–(3) are 4D-consistent [4], that is
$$T_l(u_{ijk})=T_k(u_{ijl}),\quad T_l(\psi_{ijk})=T_k(\psi_{ijl}),\quad T_l(a^{ij}_k)=T_k(a^{ij}_l).$$

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81 Hirota–Ohta system

Authors: V.E. Adler, V.V. Postnikov, 2010.08.03

$$
\begin{aligned}
-2u_t &= u_{xxx} - 3u_{xy} + 3wu_x - 3qu,\\
-2v_t &= v_{xxx} + 3v_{xy} + 3wv_x + 3qv,\\
4w_t &= w_{xxx} - 24(uv)_x + 6ww_x + 3q_y, \quad q_x = w_y.
\end{aligned} \tag{1}
$$

➢ Introduced by Hirota–Ohta [1], as the Pfaffianization of Kadomtsev–Petviashvili equation, see also [2]. The whole hierarchy can be derived also within the general approach based on Clifford algebra representations and the boson-fermion correspondence [3, 4].

➢ Bilinear form. The change of variables

$$
u = \frac{f}{h}, \quad v = \frac{g}{h}, \quad w = 2(\log h)_{xx}, \quad q = 2(\log h)_{xy} \tag{2}
$$

brings the system (1) to the form

$$
\begin{aligned}
&(2D_t - 3D_xD_y + D_x^3)f \cdot h = 0,\\
&(2D_t + 3D_xD_y + D_x^3)g \cdot h = 0,\\
&(4D_xD_t - 3D_y^2 - D_x^4)h \cdot h + 24fg = 0.
\end{aligned} \tag{3}
$$

➢ Bäcklund–Schlesinger transformation. System (1) admits the explicit auto-transformation

$$
\begin{aligned}
u_1 = u^2v + \frac{1}{2}(u_xw_x - uw_y) + \frac{w}{u}(uu_{xx} - u_x^2) + \frac{1}{4u}\big(&uu_{yy} - u_y^2\\
&-2uu_{xxy} + 2u_xu_{xy} + uu_{xxxx} - 2u_xu_{xxx} + u_{xx}^2\big),\\
v_1 = 1/u, \qquad w_1 = w + 2(\log u)_{xx}, \qquad &q_1 = q + 2(\log u)_{xy}.
\end{aligned} \tag{4}
$$

The substitutions (2) reduce the last three equations in (4) to $h_1 = f$, $g_1 = h$, that is the iterations of this mapping generate the sequence

$$
\dots \quad f = h(n_1 + 1), \quad h = h(n_1), \quad g = h(n_1 - 1) \quad \dots
$$

The system (3) takes then the form of so-called ***Pfaff lattice***

$$(2D_t - 3D_xD_y + D_x^3)h_1 \cdot h = 0, \quad (4D_xD_t - 3D_y^2 - D_x^4)h \cdot h + 24h_1h_{-1} = 0 \tag{5}$$

(introduced in [5, 6, 7, 8] within the theory of random matrix models).

➢ Bäcklund–Darboux transformation [9, 10]:

$$\begin{aligned} fh_{i,y} - f_yh_i + fh_{i,xx} - 2f_xh_{i,x} + f_{xx}h_i - 2f_ih &= 0, \\ hg_{i,y} - h_yg_i + hg_{i,xx} - 2h_xg_{i,x} + h_{xx}g_i - 2h_ig &= 0, \\ hh_{i,y} - h_yh_i - hh_{i,xx} + 2h_xh_{i,x} - h_{xx}h_i + 2fg_i &= 0; \end{aligned} \tag{6}$$

nonlinear superposition principle (2 discrete variables):

$$\begin{aligned} fh_{ij,x} - f_xh_{ij} &= f_ih_j - f_jh_i, \\ hg_{ij,x} - h_xg_{ij} &= h_ig_j - h_jg_i, \\ h_ih_{j,x} - h_{i,x}h_j &= fg_{ij} - hh_{ij}, \quad i < j; \end{aligned} \tag{7}$$

nonlinear superposition principle (3 discrete variables) [11]:

$$\begin{aligned} fh_{ijk} - f_ih_{jk} + f_jh_{ik} - f_kh_{ij} &= 0, \\ hg_{ijk} - h_ig_{jk} + h_jg_{ik} - h_kg_{ij} &= 0, \\ fg_{ijk} - h_ih_{jk} + h_jh_{ik} - h_kh_{ij} &= 0, \quad i < j < k. \end{aligned} \tag{8}$$

➢ Auxiliary linear problems [12]:

$$\psi_y = \psi_{xx} + w\psi + 2u\phi, \quad -\phi_y = \phi_{xx} + 2v\psi + w\phi;$$
$$\psi_t = \psi_{xxx} + \frac{3}{2}w\psi_x + \frac{3}{4}(w_x + q)\psi + 3u_x\phi, \quad \phi_t = \phi_{xxx} + \frac{3}{2}w\phi_x + \frac{3}{4}(w_x - q)\phi + 3v_x\psi;$$
$$\psi_1 = \psi_{xx} - \frac{u_x}{u}\psi_x + \Big(w + \frac{u_{xx} - u_y}{2u}\Big)\psi + u\phi, \quad \phi_1 = -\frac{1}{u}\psi;$$
$$\psi_x = \psi_i + w^{(i)}\psi + u\phi_i, \quad -\phi_{i,x} = \phi + v_i\psi + w^{(i)}\phi_i;$$
$$\psi_j = \psi_i + w^{(ij)}(\psi + u\phi_{ij}), \quad \phi_j = \phi_i - w^{(ij)}(v_{ij}\psi + \phi_{ij}), \quad i \le j$$

where

$$w^{(i)} = \frac{h_{i,x}}{h_i} - \frac{h_x}{h}, \quad w^{(ij)} = \frac{hh_{ij}}{h_ih_j}.$$

➢ Squared eigenfunctions constraint: see Kulish–Sklyanin system.

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82 Hirota–Satsuma equation

$$u_t = (au_{xx} + 3au^2 - 3v^2)_x, \quad v_t = -v_{xxx} - 3v_x u$$

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83 Hyperbolic equations with third order symmetries

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Here we present a complete list of nonlinear one-field hyperbolic equations that have integrable x- and y-symmetries of third order. The list includes both sine-Gordon type equations and Liouville-type equations (linearizable by differential substitutions).

In different settings, the problem of classification of some particular types of integrable hyperbolic equations had been considered in [1, 2, 3].

1. Introduction

The symmetry approach to classification of integrable PDEs (see surveys [4, 5, 6] and references there) is based on the existence of higher infinitesimal symmetries and/or conservation laws for integrable equations. This approach is especially efficient for evolution equations with one spatial variable. In particular, all integrable equations of the form

$$u_t = u_3 + F(u_2, u_1, u), \qquad u_i = \frac{\partial^i u}{\partial x^i} \tag{1}$$

were described in [7, 8]. The following list of integrable equations

List:

$$u_t = u_{xxx} + u u_x, \tag{2}$$

$$u_t = u_{xxx} + u^2 u_x, \tag{3}$$

$$u_t = u_{xxx} + u_x^2, \tag{4}$$

$$u_t = u_{xxx} - \frac{1}{2} u_x^3 + (c_1 e^{2u} + c_2 e^{-2u}) u_x, \tag{5}$$

$$u_t = u_{xxx} - \frac{3u_x u_{xx}^2}{2(u_x^2+1)} + c_1(u_x^2+1)^{3/2} + c_2 u_x^3, \tag{6}$$

$$u_t = u_{xxx} - \frac{3u_x u_{xx}^2}{2(u_x^2+1)} - \frac{3}{2}\wp(u) u_x (u_x^2+1), \tag{7}$$

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{2u_x} + \frac{3}{2u_x} - \frac{3}{2}\wp(u) u_x^3, \tag{8}$$

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{2u_x}, \tag{9}$$

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{4u_x} + c_1 u_x^{3/2} + c_2 u_x^2, \quad c_1 \neq 0 \text{ or } c_2 \neq 0, \tag{10}$$

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{4u_x} + cu, \tag{11}$$

$$\begin{aligned} u_t = u_{xxx} &- \frac{3}{4}\frac{u_{xx}^2}{u_x+1} + 3\,u_{xx} u^{-1}(\sqrt{u_x+1} - u_x - 1) \\ &- 6\,u^{-2} u_x (u_x+1)^{3/2} + 3\,u^{-2} u_x\,(u_x+1)(u_x+2), \end{aligned} \tag{12}$$

$$\begin{aligned} u_t = u_{xxx} &- \frac{3}{4}\frac{u_{xx}^2}{u_x+1} - 3\,\frac{u_{xx}\,(u_x+1)\cosh u}{\sinh u} + 3\,\frac{u_{xx}\sqrt{u_x+1}}{\sinh u} \\ &- 6\,\frac{u_x (u_x+1)^{3/2}\cosh u}{\sinh^2 u} + 3\,\frac{u_x\,(u_x+1)(u_x+2)}{\sinh^2 u} + u_x^2(u_x+3), \end{aligned} \tag{13}$$

$$u_t = u_{xxx} + 3u^2 u_{xx} + 3u^4 u_x + 9u u_x^2, \tag{14}$$

$$u_t = u_{xxx} + 3u u_{xx} + 3u^2 u_x + 3u_x^2, \tag{15}$$

$$u_t = u_{xxx}. \tag{16}$$

is equivalent to one from [7]. Here $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, and k, c, c_1, c_2, g_2, g_3 are arbitrary constants.

Remark 1. Equations (2)–(10) are integrable by the inverse scattering transform method whereas (11)–(15) are linearizable (S and C-integrable in the terminology by F. Calogero).

Remark 2. Equation (8) is equivalent to the Krichever–Novikov equation

$$u_t = u_{xxx} - \frac{3}{2}\,\frac{u_{xx}^2 + Q}{u_x}$$

up to a transformation of the form $u \to \phi(u)$. Equation (7) is equivalent to the Calogero–Degasperis equation

$$u_t = u_{xxx} - \frac{1}{2}Q''\,u_x + \frac{3}{8}\,\frac{((Q-u_x^2)_x)^2}{u_x\,(Q-u_x^2)}.$$

Here $Q = c_4u^4 + c_3u^3 + c_2u^2 + c_1u + c_0$ is an arbitrary polynomial.

The above list is complete up to transformations of the form

$$u \to \phi(u); \quad t \to t,\ x \to x + ct; \quad x \to \alpha x,\ t \to \beta t,\ u \to \lambda u; \quad u \to u + \gamma x + \delta t. \tag{17}$$

The latter transformation preserves the form (1) only for equations with $\frac{\partial F}{\partial u} = 0$. Moreover, the linear equations admit the transformation:

$$u \to u\exp(\alpha x + \beta t). \tag{18}$$

Since the symmetry approach is purely algebraic, the function ϕ and the constants $c, \alpha, \beta, \lambda, \gamma$ and δ supposed to be complex-valued. Thus, for example, we do not distinguish between equations $u_t = u_{xxx} - u_x^3$ and $u_t = u_{xxx} + u_x^3$.

For scalar hyperbolic equations of the form

$$u_{xy} = \Psi(u, u_x, u_y) \tag{19}$$

the symmetry approach postulates the existence of both x-symmetries

$$u_t = A(u, u_x, u_{xx}, \ldots,), \tag{20}$$

and y-symmetries

$$u_\tau = B(u, u_y, u_{yy}, \dots,). \tag{21}$$

Two equations (19) are called *equivalent* if they are related by transformations of the form

$$x \leftrightarrow y; \quad u \to \phi(u); \quad x \to \alpha x, \;\; y \to \beta y, \;\; u \to \lambda u; \quad u \to u + \gamma x + \delta y. \tag{22}$$

Here, in general, the function ϕ and the constants are supposed to be complex-valued. For linear equations (19) the transformations

$$u \to u \exp(\alpha x + \beta y); \qquad u \to u + c\, x\, y \tag{23}$$

are also allowed.

For the well-known integrable sin-Gordon[2] equation

$$u_{xy} = c_1 e^u + c_2 e^{-u} \tag{24}$$

the simplest x and y-symmetries are given by

$$u_t = u_{xxx} - \frac{1}{2} u_x^3, \qquad u_\tau = u_{yyy} - \frac{1}{2} u_y^3.$$

These evolution equations are integrable themselves (a special case of equation (5)).

The general higher symmetry classification for equations (19) turns out to be a very complicated problem, which has not been solved so far. Some important special results have been obtained in [9, 10, 11]. In general, all three functions Ψ, A, B should be found from the compatibility conditions for equations (19), (20), and (21). However, if the functions A and B are somehow fixed, then it is not difficult to verify whether the corresponding function Ψ exists and to find it.

To describe all integrable equations (19) of the sin-Gordon type, we assume (see the section Discussion) that symmetries (20) and (21) are *integrable* evolution equations of the form

$$u_t = u_{xxx} + F(u, u_x, u_{xx}), \qquad u_\tau = u_{yyy} + G(u, u_y, u_{yy}). \tag{25}$$

---

[2] We do not distinguish between sin-Gordon and sinh-Gordon equations

We take equations from List 1 one by one as x-symmetry and find all equations (19) having this symmetry. After that in each case we find the corresponding y-symmetry or verify that it do not exist. In Section 2 we present all hyperbolic equations with x- and y-symmetries (25) thus obtained.

Integrable hyperbolic equations can be separated in accordance to presence or absence of x and y-integrals (see the section Discussion). Consider, for instance, the Liouville equation

$$u_{xy} = e^u.$$

It is easy to verify that the function

$$P = u_{xx} - \frac{1}{2}u_x^2$$

does not depend on y (i.e. is a function depending on x only) for any solution $u(x, y)$ of the Liouville equation. Analogously, the function

$$Q = u_{yy} - \frac{1}{2}u_y^2$$

does not depend on x.

A function $w(x, y, u, u_y, u_{yy}, \ldots)$ that does not depend on x on any solution of (19) is called *x-integral*. The y-integrals are defined similarly. An equation of the form (19) is called *equation of the Liouville type* (or Darboux integrable equation), if the equation possesses both nontrivial x- and y-integrals. Some of the integrable hyperbolic equations found in Section 2 are equations of the Liouville type. The general classification problem for Liouville type equations was considered in [11].

In contrast to the Liouville equation, the sin-Gordon equation (24) has no x- or y-integrals for non-zero values of the constants c_i. There are two types of such equations. Equations of the first type can be reduced to the linear Klein–Gordon equation $u_{xy} = cu$ by differential substitutions. If an equation with the third order symmetries has no integrals and does not admit linearizing substitutions, we call it *equation of sin-Gordon type*. Such equations are integrable by the inverse scattering method. The following equations from the list of Section 2 are equations of such kind:

$$u_{xy} = c_1 e^u + c_2 e^{-u}; \tag{26}$$

$$u_{xy} = f(u)\sqrt{u_x^2 + 1}, \quad f'' = cf; \tag{27}$$

$$u_{xy} = \sqrt{u_x}\sqrt{u_y^2+1}; \tag{28}$$

$$u_{xy} = \sqrt{\wp(u)-\mu}\,\sqrt{u_x^2+1}\sqrt{u_y^2+1}. \tag{29}$$

Here $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, $4\mu^3 - g_2\mu - g_3 = 0$ and $c, c_1, c_2, a, \mu, g_2, g_3$ are constants. Equations (27), (28), (29) are related to equation (26) via differential substitutions [11, 14, 13].

2. Hyperbolic equations with third order symmetries

Theorem 3. *Suppose both x- and y-symmetry of a hyperbolic equation of the form (19) belong to the list (2)–(16) up to transformations (17), (18). Then this equation belongs to the following list:*

$$u_{xy} = f(u)\sqrt{u_x^2+1}, \qquad f'' = cf, \tag{30}$$

$$u_{xy} = ae^u + be^{-u}, \tag{31}$$

$$u_{xy} = \sqrt{u_x}\sqrt{u_y^2+1}, \tag{32}$$

$$u_{xy} = \sqrt{u_x^2+1}\sqrt{u_y^2+1}, \tag{33}$$

$$u_{xy} = \sqrt{\wp(u)-\mu}\sqrt{u_x^2+1}\sqrt{u_y^2+a}, \tag{34}$$

$$u_{xy} = 2uu_x, \tag{35}$$

$$u_{xy} = 2u_x\sqrt{u_y}, \tag{36}$$

$$u_{xy} = u_x\sqrt{u_y^2+1}. \tag{37}$$

$$u_{xy} = \sqrt{u_xu_y}, \tag{38}$$

$$u_{xy} = \frac{u_x(u_y+a)}{u}, \quad a \neq 0, \tag{39}$$

$$u_{xy} = (ae^u + be^{-u})u_x, \tag{40}$$

$$u_{xy} = u_y\eta\,\sinh^{-1}u\big(\eta\, e^u - 1\big), \tag{41}$$

$$u_{xy} = \frac{2u_y\eta}{\sinh u}(\eta\cosh u - 1), \tag{42}$$

$$u_{xy} = \frac{2\xi\eta}{\sinh u}\big((\xi\eta + 1)\cosh u - \xi - \eta\big), \tag{43}$$

$$u_{xy} = u^{-1}u_y\,\eta\,(\eta - 1) + c\,u\,\eta\,(\eta + 1), \tag{44}$$

$$u_{xy} = 2u^{-1}u_y\,\eta\,(\eta - 1), \tag{45}$$

$$u_{xy} = 2u^{-1}\xi\,\eta\,(\xi - 1)(\eta - 1), \tag{46}$$

$$u_{xy} = u^{-1}u_xu_y - 2u^2u_y, \tag{47}$$

$$u_{xy} = u^{-1}u_x(u_y + a) - uu_y \tag{48}$$

$$u_{xy} = \sqrt{u_y} + au_y, \tag{49}$$

$$u_{xy} = cu, \tag{50}$$

up to transformations (22), (23). Here $\wp$ *is the Weierstrass function:* $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$*;* $\xi = \sqrt{u_y + 1}$*,* $\eta = \sqrt{u_x + 1}$*;* a, b, c, g_2, g_3 *are arbitrary constants;* μ *is a root of the equation* $4\mu^3 - g_2\mu - g_3 = 0$.

Proof. If (25) is an x-symmetry for (19), then

$$\frac{d^2}{dxdy}(u_{xxx} + F) = \frac{\partial\Psi}{\partial u_x}\frac{d}{dx}(u_{xxx} + F) + \frac{\partial\Psi}{\partial u_y}\frac{d}{dy}(u_{xxx} + F) + \frac{\partial\Psi}{\partial u}(u_{xxx} + F). \tag{51}$$

Eliminating all mixed derivatives in virtue of (19), we arrive at a defining relation, which has to be fulfilled identically with respect to the variables u, u_y, u_x, u_{xx}, u_{xxx}. Comparing the coefficients at u_{xxx} in this relation, we get

$$\frac{d}{dy}\frac{\partial F}{\partial u_{xx}} + 3\frac{d}{dx}\frac{\partial\Psi}{\partial u_x} = 0. \tag{52}$$

If some equation from the list (2)–(16) is taken for the x-symmetry then the function F is known and the defining relation can also be split with respect to u_{xx}.

For example, let equation (7) be an x-symmetry for (19). Then the u_{xx}-splitting of (52) gives rise to:

$$(u_x^2+1)^2\frac{\partial^2\Psi}{\partial u_x^2}-u_x(u_x^2+1)\frac{\partial\Psi}{\partial u_x}+(u_x^2-1)\Psi=0,$$
$$(u_x^2+1)\left(\Psi\frac{\partial^2\Psi}{\partial u_x\partial u_y}+u_x\frac{\partial^2\Psi}{\partial u\partial u_x}\right)-u_x^2\frac{\partial\Psi}{\partial u}-u_x\Psi\frac{\partial\Psi}{\partial u_y}=0.$$

The general solution of this system is given by

$$\Psi=\sqrt{u_x^2+1}\left(g(u,u_y)+C\ln(u_x+\sqrt{u_x^2+1})\right).$$

Substituting this expression into (51) and finding the coefficient at u_{xx}^3, we obtain $C=0$ and therefore

$$\Psi=g(u,u_y)\sqrt{u_x^2+1}. \tag{53}$$

Splitting (51) with respect to u_{xx} and u_x, we obtain that (51) is equivalent to a system consisting of (53) and equations

$$g\frac{\partial^2 g}{\partial u\partial u_y}-\frac{\partial g}{\partial u}\frac{\partial g}{\partial u_y}=0,\qquad \wp'(u)u_y=2\frac{\partial g}{\partial u}\frac{\partial g}{\partial u_y},$$
$$g^2\frac{\partial^2 g}{\partial u_y^2}+g\left(\frac{\partial g}{\partial u_y}\right)^2-3g\wp+\frac{\partial^2 g}{\partial u^2}=0, \tag{54}$$

where $(\wp')^2=4\wp^3-g_2\wp-g_3$. Since $\wp'\neq 0$ we have $g_u\neq 0$ and $g_{u_y}\neq 0$. It follows from the first two equations (54) that $g=\sqrt{\wp(u)-\mu}\sqrt{u_y^2+a}$, where μ and a are constants of integration. The third equation is equivalent to the algebraic equation $4\mu^3-g_2\mu-g_3=0$ for μ. Thus, we get equation (34).

To prove Theorem 1 we perform similar computations for each equation from the list (2)–(15) taken for x-symmetry. For equations (2), (4), and (8) the corresponding hyperbolic equation does not exist. In contrast, equation (12) is an x-symmetry for several different hyperbolic equations. Indeed, in this case calculating the coefficient at u_{xx} in (52), we get

$$2(u_x+1)^2\frac{\partial^2\Psi}{\partial u_x^2}-(u_x+1)\frac{\partial\Psi}{\partial u_x}+\Psi=0,$$

which implies $\Psi = f_1(u, u_y)(u_x + 1) + f_2(u, u_y)\sqrt{u_x + 1}$. Substituting this into (52), we obtain

$$\left(u\frac{\partial f_1}{\partial u_y} - 1\right)\left(u^2 f_1\frac{\partial f_1}{\partial u_y} - 3uf_1 + 2u_y\right) = 0,$$
$$u^2\left(f_1\frac{\partial f_1}{\partial u_y} + \frac{\partial f_1}{\partial u}\right) - 2uf_1 + 2u_y = 0,$$
$$f_2 = 2f_1 - \frac{2}{u}u_y - uf_1\frac{\partial f_1}{\partial u_y}.$$

If the first factor in the first equation is equal to zero, we arrive at (44). If the second factor equals zero, then we find that

$$f_1 = \frac{2u_y\sqrt{au_y + 1}}{u(1 + \sqrt{au_y + 1})},$$

where a is a constant. The case $a \neq 0$ corresponds to (46), while $a = 0$ leads to equation (44) with $c = 0$. The limit $a \to \infty$ gives us equation (45).

The computations for remaining x-symmetries from the list except for the Swartz-KdV equation (9) are very similar and we do not display them here.

Consider the Swartz-KdV equation (9). This equation is exceptional because there is a wide class of hyperbolic equations with x-symmetry (9). We find all equations from this class that have y-symmetries.

It is easy to verify that equation

$$u_{xy} = f(u, u_y)u_x \tag{55}$$

has the following symmetry

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{2u_x} + q(u)u_x^3, \tag{56}$$

where

$$\left(\frac{\partial}{\partial u} + f\frac{\partial}{\partial u_y}\right)^2 f + 2qf + q'u_y = 0. \tag{57}$$

The function $q(u)$ can be normalized by an appropriate transformation $u \to \varphi(u)$, but we prefer to use such transformations for bringing the y-symmetry to one of equations (2)–(15). Here and in the sequel we have in mind the transformation $y \to x, \tau \to t$ in the second formula from (25).

Any of y-symmetries has the form

$$u_t = u_3 + A_2(u, u_1)u_2^2 + A_1(u, u_1)u_2 + A_0(u, u_1), \qquad u_n = \frac{\partial^n u}{\partial u_y^n}.$$

Equation (52) under $x \leftrightarrow y$ is equivalent to

$$\begin{aligned} &3\frac{\partial^2 f}{\partial u_y^2} + 2\frac{\partial(fA_2)}{\partial u_y} + 2\frac{\partial A_2}{\partial u} = 0, \\ &3u_y\frac{\partial^2 f}{\partial u \partial u_y} + 3f\frac{\partial f}{\partial u_y} + 2A_2 u_y\frac{\partial f}{\partial u} + f\frac{\partial A_1}{\partial u_y} + 2A_2 f^2 + \frac{\partial A_1}{\partial u} = 0. \end{aligned} \tag{58}$$

Equations (57) and (51) give rise to additional restrictions for the functions f and q.

For symmetries (2)–(5) we have $A_1 = A_2 = 0$ and equations (58) imply $f = u_y g(u) + h(u)$, $gh = 0$, $g' + g^2 = 0$. In the case $g \neq 0$ we get (39) with $a = 0$. For $g = 0$ it follows from (57) that $q' = 0$ and $f'' + 2qf = 0$. If $q \neq 0$, then without loss of generality we take $q = -\frac{1}{2}$ and arrive at equation (40). In the case $g = 0, q = 0$ we get equation (35).

For symmetries (6) and (7) we have $A_1 = 0, A_2 = -3/2\, u_y(u_y^2 + 1)^{-1}$. It follows from (57) and (58) that $f = h(u)u_x\sqrt{u_y^2 + 1}\,, h'' = 2h(h^2 + c_0), q = c_0 - 3/2\, h^2$. If $h' = 0$, then we put $h = 1$ and obtain equation (37). In the case $h' \neq 0$ we get $h = \sqrt{\wp - \mu}$, $q = -3/2\wp$. The corresponding hyperbolic equation is given by (34) with $x \leftrightarrow y$ and $a = 0$.

For symmetries (8), (9) $A_2 = -\frac{3}{2}u_y^{-1}$, $A_1 = 0$. It follows from (58) that $f = g(u)u_y$. So, we obtain the equation $u_{xy} = g(u)u_x u_y$. Both x- and y-symmetries of the equation have the form (56), where

$$q = C\exp\left(-2\int g(u)\,du\right) - g' - \frac{1}{2}g^2.$$

The equation can be reduced to the d' Alembert equation $u_{xy} = 0$ by the transformation

$$\bar{u} = \int du \exp\left(-\int g(u)\, du\right).$$

For symmetries (10) and (11) $A_2 = -\frac{3}{4}u_y^{-1}$, $A_1 = 0$. It follows from (58), (57) that $f = g(u)u_y + C\sqrt{u_y}$, $gC = 0$, $qC = 0$, $g' + g^2 = 0$, $q' + 2qg = 0$. If $C \neq 0$, then $q = g = 0$. Taking $C = 2$, we get (36). If $C = 0$, then $g = u^{-1}, q = c_0 u^{-2}$, and we arrive at (39) with $a = 0$.

If the y-symmetry has the form (12), then it follows from (57), (58) that $f = ku^{-1}(u_y + 1 - \sqrt{u_y + 1})$, $(k-1)(k-2) = 0$, $q = 3(2-k)/(8u^2)$. If $k = 1$ we get (44) up to $x \leftrightarrow y$. The case $k = 2$ leads to (45).

In the case of y-symmetry (13) the system of equations (57), (58) has two solutions corresponding to equations (41), (42) with $x \leftrightarrow y$.

Symmetry (14) gives rise to equation (47) up to $x \leftrightarrow y$.

Symmetry (15) corresponds to the following equation

$$u_{xy} = \frac{u_x u_y}{u + a} - (u + a)u_x.$$

The shift $u \to u - a$ brings it to a special case of equation (48).

Considering the linear x-symmetry (16), we obtain equation (39) with an arbitrary parameter a, equation (50), and

$$u_{xy} = a\, u_x + f(u_y - a\, u), \tag{59}$$

where f satisfies some nonlinear third order ODE. The requirement of existence of a y-symmetry leads to (49). ■

More detailed information on each equation from the list (30)–(50) can be found in Appendix 1.

3. Discussion

The hyperbolic equations of the form (19) that have both x and y-integrals were described in [11]. In particular, it was shown that any such equation possesses both x and y higher symmetries depending on arbitrary functions. Although not all of these symmetries are integrable evolution equation, there exist integrable symmetries among of them.

There are integrable equations having only y-integrals (or only x-integrals). An example of such equation is given by (34) with $a = 0$. Namely, the equation

$$u_{xy} = \xi'(u) u_y \sqrt{u_x^2 + 1}, \tag{60}$$

where $\xi'(u) = \sqrt{\wp - \mu}$, has the following first order y-integral

$$I = (u_x + \sqrt{u_x^2 + 1}\,)\, e^{-\xi}$$

and has no x-integrals for non-degenerate Weierstrass function $\wp$. Notice that the same formula gives a y-integral for (60) with arbitrary function ξ.

In some sense equations (19) having integrals can be reduced to ODEs. If we are looking for equations (19) integrable by the inverse scattering transform method, we should concentrate on integrable equations (19) without integrals. There are two classes of such equations. The first one consists of the Klein–Gordon equation
$u_{xy} = cu$, $c \neq 0$ and equations related to it via differential substitutions. The symmetries for such equations are C-integrable in Calogero's terminology.

The second class of hyperbolic integrable equations having no integrals contains equations that cannot be reduced to a linear form by differential substitutions. This most interesting class consists of equations admitting only S-integrable higher symmetries. Such equations can be regarded as S-*integrable* hyperbolic equations.

At first glance the anzats (25) seems to be very restrictive if we want to describe all S-integrable equations (19). The first question is: why are only third order equations taken for symmetries? We can justify this in the following way. All known S-integrable hierarchies of evolution equations (20) contain either a third order or a fifth order equation. For polynomial hierarchies this is not an observation but a rigorous statement [12]. That is why it is enough to consider hyperbolic equations with symmetries of the third order (sin-Gordon type equations) and hyperbolic equations with fifth order symmetries (Tzitzeica type equations). The following Tzitzeica-type S-integrable equations are known up until now [11, 15]:

$$u_{xy} = c_1 e^u + c_2 e^{-2u}, \tag{61}$$

$$u_{xy} = S(u) f(u_x) g(u_y), \tag{62}$$
$$u_{xy} = h(u)\, g(u_y), \quad h'' = 0, \tag{63}$$

where

$$(f + 2u_x)^2 (u_x - f) = 1, \qquad (g + 2u_y)^2 (u_y - g) = 1,$$
$$(S' - 2S^2)^2 (S' + S^2) = c_1, \qquad \omega'^2 = 4\omega^3 + c^2.$$

We are planning to consider the Tzitzeica type equations in a separate paper.

The second question is: why do we restrict ourselves by symmetries $u_t = u_{xxx} + F(u, u_x, u_{xx})$ instead of general symmetries of the form

$$u_t = \Phi(u, u_x, u_{xx}, u_{xxx})\,? \tag{64}$$

The main reason is the following statement (see [16]). Suppose that equation (64) is a symmetry for equation (19), then

$$\frac{d}{dy}\Big(\frac{\partial \Phi(u, u_x, u_{xx}, u_{xxx})}{\partial u_{xxx}}\Big) = 0.$$

Therefore, if we assume that (19) has no nontrivial integrals, then

$$\frac{\partial \Phi(u, u_x, u_{xx}, u_{xxx})}{\partial u_{xxx}} = const.$$

.

Appendix 1. Symmetries, integrals and differential substitutions

Here we give more information on equations from the list (30)–(50). Integrable third order symmetries, x-integrals $J(u, u_y, u_{yy}, \dots)$, y-integrals $I(u, u_x, u_{xx}, \dots)$ and, in some cases, general solutions are presented.

Equation (30). The symmetries have the following form:

$$u_t = u_{xxx} - \frac{c}{2} u_x^3 - \frac{3}{2} f^2(u) u_x, \quad u_t = u_{yyy} - \frac{3 u_y u_{yy}^2}{2(u_y^2 + 1)} - \frac{c}{2} u_y^3.$$

The formula (30) describes two non-equaivalent sin-Gordon type equations:
(30a). $u_{xy} = u\sqrt{u_x^2 + 1}$; **(30b).** $u_{xy} = \sin u\sqrt{u_x^2 + 1}$

and two Liouville type equations:
(30c). $u_{xy} = \sqrt{u_x^2 + 1}$; the integrals are:

$$I = \frac{u_{xx}}{\sqrt{u_x^2 + 1}}, \quad J = u_{yy} - u;$$

the linearizing substitution $u_x = \sinh(y + v_x)$ reduces the equation to the linear one: ???. The general solution is given by:

$$u = \int \sinh(y + f(x))\, dx + g(y);$$

(30d). $u_{xy} = e^u\sqrt{u_x^2 + 1}$; the integrals are:

$$I = \frac{u_{xx}}{\sqrt{u_x^2 + 1}} - \sqrt{u_x^2 + 1}, \quad J = u_{yy} - \frac{1}{2}u_y^2 - \frac{1}{2}e^{2u}.$$

The general solution is given by

$$u(x, y) = \ln\left(\frac{-\varphi(x)g'(y)}{\big(g(y) + h(x)\big)\big(\varphi(x) + f(x)\big(g(y) + h(x)\big)\big)}\right),$$

$$\varphi(x) = \exp\left(\int \frac{f(x)}{4f'(x)}\, dx\right), \quad h(x) = \int \frac{f'(x)\varphi(x)}{f^2(x)}\, dx.$$

Equation (31). Both x- and y-symmetries have the form (5), where $c_1 = c_2 = 0$. If $ab \neq 0$, then we have the *sin*-Gordon equation. There is the following degeneration:

(31a). $u_{xy} = e^u$ is the Liouville equation. Its integrable symmetries have the same form as for the *sin*-Gordon equation. The integrals were shown in the Introduction. The general solution

$$u(x, y) = \log\left(\frac{2f'(x)g'(y)}{(f(x) + g(y))^2}\right)$$

was found by Liouville in 1853.

Equation (32). The x-symmetry has the form (10), where $c_1 = 0, c_2 = -3/4$; the y-symmetry is of the form (6), where $c_1 = c_2 = 0$. It is an S-integrable equation.

Equation (33). Both x- and y-symmetries have the form (6), where $c_1 = 0, c_2 = -1/2$. It is an S-integrable equation.

Equation (34). The x-symmetry is of the form (7), the form of the y-symmetry is analogous:

$$u_\tau = u_{yyy} - \frac{3u_y u_{yy}^2}{2(u_y^2 + a)} - \frac{3}{2}\wp(u)u_y(u_y^2 + a).$$

If $a = 0$, then this symmetry is equivalent to (9).

In the general case the equation can be rewritten using the Jacobi function sn as:

$$u_{xy} = \frac{1}{\operatorname{sn}(u,k)}\sqrt{u_x^2 + 1}\sqrt{u_y^2 + a}. \tag{65}$$

This is an S-integrable equation except for the degenerate cases considered below. Notice that the formulas

$$\sqrt{\wp(u, g_2, g_3) - \mu_1} = \frac{\operatorname{cn}(u,k)}{\operatorname{sn}(u,k)}, \quad \sqrt{\wp(u, g_2, g_3) - \mu_2} = \frac{\operatorname{cn}(u,k)}{\operatorname{dn}(u,k)}$$

lead to another forms of equation (65). They look different but can be reduced to (65) by substitutions of the form $(u,k) \to (\lambda u, f(k))$ (see [17], Sec. 13.22).

There are two degenerations of the Weierstrass function. In the first case when $\wp(u) = u^{-2}$ we have $\mu = 0$ and $\sqrt{\wp - \mu} = u^{-1}$. In the second case $\wp(u) = \sin^{-2} u - \frac{1}{3}$, $\mu = -\frac{1}{3}$ and $\sqrt{\wp - \mu} = \sin^{-1} u$.

(34a). Equation $u_{xy} = u^{-1}\sqrt{u_x^2 + 1}\sqrt{u_y^2 + a}$ is C-integrable, the integrals are:

$$I = \frac{u_{xx}}{\sqrt{u_x^2 + 1}} + \frac{1}{u}\sqrt{u_x^2 + 1}, \quad J = \frac{u_{yy}}{\sqrt{u_y^2 + a}} + \frac{1}{u}\sqrt{u_y^2 + a}.$$

The general solution is given by:

$$u(x,y) = \sqrt{f(x) + g(y)}\left(-\int \frac{dx}{f'(x)} - a\int \frac{dx}{g'(y)}\right)^{1/2}.$$

(34b). Equation $u_{xy} = (\sin u)^{-1}\sqrt{u_x^2+1}\sqrt{u_y^2+a}$ is C-integrable, the integrals are:

$$I = \frac{u_{xx}}{\sqrt{u_x^2+1}} + \cot u\sqrt{u_x^2+1}, \quad J = \frac{u_{yy}}{\sqrt{u_y^2+a}} + \cot u\sqrt{u_y^2+a}.$$

If $a = 0$, then the general solution is

$$u(x,y) = 2\arccos\left(\frac{f(x)+h(x)+g(y)}{2f(x)}\right)^{1/2}, \quad h(x) = \int \sqrt{f'^2 - f^2}\,dx.$$

If $a \neq 0$, then the general solution is given by

$$\begin{aligned} &u(x,y) = \arccos\Psi(x,y),\\ &\Psi(x,y) = \frac{1}{2}w(x)\left[e^g(\xi+h)^2 - e^{-g}\right](2w' + fw) + (\xi+h)e^g, \quad g = g(y),\\ &h(y) = \int e^{-g}\sqrt{g'^2 - a}\,dy, \quad f'(x) = \frac{1}{2}(1+f^2) - 2\frac{w''}{w}, \quad \xi(x) = \int \frac{dx}{w^2(x)}. \end{aligned}$$

(34c). $a = 0$, $u_{xy} = f(u)u_y\sqrt{u_x^2+1}$. There exists the following y-integral

$$I = (u_x + \sqrt{u_x^2+1})\exp(-\xi(u)), \quad \xi(u) = \int f(u)\,du.$$

The integration with respect to y leads to the following ODE:

$$u_x = \frac{1}{2}\left(h(x)e^{\xi} - \left(h(x)e^{\xi}\right)^{-1}\right).$$

All remaining equations are C-integrable. Some of them have two integrals and can be integrated in a closed form. Others have no integrals and can be reduced to the linear Klein–Gordon equation.

Equation (35). The x-symmetry has the form (9) and the y-symmetry is the mKdV equation $u_\tau = u_{yyy} - 6u^2u_y$. The integrals are:

$$I = \frac{u_{xxx}}{u_x} - \frac{3u_{xx}^2}{2u_x^2}, \quad J = u_y - u^2.$$

The general solution is given by

$$u(x,y)=\frac{g''(y)}{2g'(y)}-\frac{g'(y)}{f(x)+g(y)}.$$

Equation (36). The x-symmetry has the form (9) and the y-symmetry is (10), where $c_1=0, c_2=-3$. The integrals are:

$$I=\frac{u_{xxx}}{u_x}-\frac{3u_{xx}^2}{2u_x^2},\qquad J=\sqrt{u_y}-u.$$

The general solution is given by

$$u(x,y)=-\frac{g'(y)}{f(x)+g(y)}+\int\frac{(g'')^2}{4g'^2}\,dy.$$

Equation (37). The y-symmetry has the form (6), where $c_1=0, c_2=-1/2$ and the x-symmetry is

$$u_t=u_{xxx}-\frac{3u_{xx}^2}{2u_x}-\frac{1}{2}u_x^3.$$

This symmetry can be reduced to (9) by $u\to\ln u$. The integrals are:

$$I=\frac{u_{xxx}}{u_x}-\frac{3u_{xx}^2}{2u_x^2}-\frac{1}{2}u_x^2,\qquad J=(u_y+\sqrt{u_y^2+1}\,)e^{-u}.$$

The general solution is given by

$$u(x,y)=\ln\left[1+\frac{g(y)}{f(x)+h(y)}\right]+\int g^{-1}\sqrt{g'^2-g^2}\,dy,\qquad h=-\frac{1}{2}g-\frac{1}{2}\int\sqrt{g'^2-g^2}\,dy.$$

Equation (38) (the Goursat equation). Both x- and y-symmetries have the form (11) with arbitrary constant c.

The equation is reduced to the Klein–Gordon equation $v_{xy}=\frac{1}{4}v$ by any of the following two differential substitutions:

$$(1)\ u_x=4v_x^2,\quad u_y=v^2;\qquad (2)\ u_x=v^2,\quad u_y=4v_y^2.$$

Equation (39). The x-symmetry has the form (11), where $c=0$ and the y-symmetry can be obtained from (5) by the substitution $c_2=0$, $u\to -\ln u$. Moreover, there exists the following second order y-symmetry $u_\tau=u_{yy}-2u^{-1}(u_y^2+au_y)$.

The integrals and the general solution are:

$$I=\frac{u_{xx}}{u_x},\qquad J=\frac{u_y+a}{u};\qquad u(x,y)=\frac{f(x)-ag(y)}{g'(y)}.$$

Equation (40). The x-symmetry has the form (56), where $q=-\frac{1}{2}$ and the y-symmetry is given by (5), where $c_1=-\frac{3}{2}a^2, c_2=-\frac{3}{2}b^2$. The integrals are:

$$I=\frac{u_{xxx}}{u_x}-\frac{3u_{xx}^2}{2u_x^2}-\frac{1}{2}u_x^2,\qquad J=u_y-ae^u+be^{-u}.$$

In the case $a\neq 0$ the general solution is given by

$$u(x,y)=\ln g(y)+\ln\left[1+\frac{h(y)}{f(x)-a\varphi(y)}\right],\qquad \ln h=\int(ag+bg^{-1})\,dy,\qquad \varphi=\int gh\,dy;$$

if $a=0$ then

$$u(x,y)=\ln\frac{f(x)-bg(y)}{g'(y)}.$$

Equation (41). The x-symmetry has the form (13). There are the following y-symmetries:

$$u_t=u_{yyy}-\frac{3}{2}(3+\coth u)u_yu_{yy}+\frac{1}{4}(3\coth^2 u+6\coth u+7)u_y^3,$$
$$u_t=u_{yy}-\frac{1}{2}(3+\coth u)u_y^2.$$

The integrals are:

$$I=\frac{e^{-u}\eta^2-2\eta+e^u}{\sinh u},\qquad J=\frac{u_{yy}}{u_y}-\frac{1}{2}u_y(\coth u+3).$$

The general solution is given by:

$$u(x,y) = -\frac{1}{2}\ln(1+\psi^2), \quad \psi = f(x)(g(y)+h(x)), \quad f' = f - \frac{1}{4}f^3 h'^2.$$

Equation (42). The x-symmetry has the form (13). There are the following y-symmetries:

$$u_t = u_{yyy} - 6u_y u_{yy}\coth u + 2(3\coth^2 u - 1)u_y^3, \quad u_t = u_{yy} - 2u_y^2 \coth u.$$

The integrals are:

$$I = \frac{\eta - e^u}{\eta - e^{-u}}, \quad J = \frac{u_{yy}}{u_y} - 2u_y \coth u.$$

The general solution is:

$$u(x,y) = \frac{1}{2}\ln\left|\frac{\psi+1}{\psi-1}\right|, \quad \psi = f(x)(g(y)+h(x)), \quad h' = -\frac{f'^2 + 4f^2}{4f^3}.$$

Equation (43). Both x- and y-symmetries have the form (13). The equation is reduced to the Klein–Gordon equation $v_{xy} = v$ by the following differential substitution:

$$u_x = \left(v^{-1}v_x \sinh u + \cosh u\right)^2 - 1, \quad u_y = \left(v^{-1}v_y \sinh u + \cosh u\right)^2 - 1.$$

Equation (44). There are x-symmetry of the form (12) and the following y-symmetry:

$$u_\tau = u_{yyy} - \frac{3u_y u_{yy}}{2u} + \frac{3u_y^3}{4u^2} - \frac{3c}{4}(2uu_{yy} + 2u_y^2 - cu^2 u_y).$$

The equation can be reduced to the Klein–Gordon equation $v_{xy} = cv$ by the following differential substitution:

$$u = v^2/z, \quad z_x = -v_x^2, \; z_y = -cv^2.$$

If $c = 0$ then the Klein–Gordon equation is reduced to the d'Alembert equation and the following two integrals appear:

$$I = \frac{(\eta-1)^2}{u}, \quad J = \frac{u_{yy}}{u_y} - \frac{u_y}{2u}.$$

The general solution is:

$$u(x,y) = \frac{(f(x)+g(y))^2}{z(x)}, \quad z(x) = -\int f'^2(x)\,dx.$$

Notice that if $c = 0$ the equation admits a second order symmetry.

Equation (45). There are x-symmetry of the form (12) and the following two y-symmetries:

$$u_\tau = u_{yyy} - 6u^{-1}u_y u_{yy} + 6u^{-2}u_y^3, \qquad u_\tau = u_{yy} - 2u^{-1}u_y^2.$$

The integrals and the general solution are given by:

$$I = \frac{\eta - 1}{u}, \quad J = \frac{u_{yy}}{u_y} - 2\frac{u_y}{u}; \quad u(x,y) = \frac{f^2(x)}{h(x)+g(y)}, \quad h(x) = -\int f'^2(x)\,dx.$$

Equation (46). Both x- and y-symmetries have the form (12). The integrals are of the form:

$$I = \frac{u_{xx}}{\eta(\eta-1)} - \frac{2}{u}\eta(\eta-1), \quad J = \frac{u_{yy}}{\xi(\xi-1)} - \frac{2}{u}\xi(\xi-1).$$

The general solution is given by:

$$u(x,y) = \frac{(f(x)+g(y))^2}{z(x,y)}, \quad z(x,y) = -\int f'^2(x)\,dx - \int g'^2(y)\,dy.$$

Equation (47). There are x-symmetry of the form (14) and the following two y-symmetries:

$$u_\tau = u_{yyy} - 9u^{-1}u_y u_{yy} + 12u^{-2}u_y^3, \qquad u_\tau = u_{yy} - 3u^{-1}u_y^2.$$

The integrals are of the form:

$$I = \frac{u_x}{u} + u^2, \quad J = \frac{u_{yy}}{u_y} - 3\frac{u_y}{u}.$$

The general solution is:

$$u(x,y) = \left(\frac{f'(x)}{2\big(f(x)+g(y)\big)}\right)^{1/2}.$$

Equation (48). There are x-symmetry of the form (15) and the following two y-symmetries:

$$u_\tau = u_{yyy} - 3u^{-1}(2u_y + a)u_{yy} + 3au^{-2}u_y(3u_y + a) + 6u^{-2}u_y^3, \quad u_\tau = u_{yy} - 2u^{-1}u_y(u_y + a).$$

When $a = 0$ the y-symmetry (9) is also admitted. The equation can be reduced to the Klein–Gordon equation $v_{xy} = -av$ by the following substitution:

$$u_x = \left(\frac{v_x}{v} - u\right)(u - \lambda), \quad u_y = \frac{1}{\lambda}\left(u\frac{v_y}{v} + a\right)(u - \lambda),$$

where λ is arbitrary parameter. If $a = 0$, then the Klein–Gordon equation is reduced to the d'Alembert equation and the following two integrals appear:

$$I = \frac{u_x}{u} + u, \qquad J = \frac{u_{yy}}{u_y} - 2\frac{u_y}{u}.$$

In this case there exists the general solution of the form $u(x, y) = f'(x)(f(x) + g(y))^{-1}$.

Equation (49). The x-symmetry is $u_t = u_{xxx} - \frac{3}{2}\,a\,u_{xx}$ and the y-symmetry has the form (11), where $c = 0$ and x is replaced by y. The integrals are of the form:

$$I = u_{xxx} - \frac{3}{2}\,a\,u_{xx} + \frac{a^2}{2}\,u_x, \quad J = \frac{u_{yy}}{a\,u_y + \sqrt{u_y}}.$$

The general solution is given by:

$$u(x, y) = f(x) + e^{ax}\int\left(g(y) + \frac{1 - e^{-ax/2}}{a}\right)dy.$$

The limit $a \to 0$ is admited here.

Equation (50). There are infinitely many symmetries of the form $u_t = P(\partial_x, \partial_y)u$, where P is an arbitrary polynomial with constant coefficients. In particular, there exist x- and y-symmetries of the form $u_t = P_1(\partial_x)u$ and $u_t = P_2(\partial_y)u$. If $c \neq 0$ integrals do not exist otherwise the simplest integrals are: $I = u_x$, $J = u_y$.

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84 Integrability

Integrable equations can be divided into linearizable ones and equations integrable by inverse scattering transform method (C- and S-integrable equations accordingly to Calogero). The following rigorous definition is formulated in terms of the canonical series of the conservation laws.

Definition 1. If the canonical series for an evolutionary equation admitting the formal symmetry contains the conservation laws of the unbounded order then the equation is called S-integrable, otherwise it is called C-integrable.

It is important to notice that existence of an infinite sequence of conservation laws does not equivalent to S-integrability. In the example of the linear equation $u_t = u_3$ the function u_n^2 is the density of conservation law for all $n = 1, 2, \dots$. However, the formal symmetry for this equation is D_x and all canonical conservation laws are trivial.

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85 Integrable discretization

The problem of finding a discretization which preserves the integrability property is one of the central ones in the theory of integrable dynamical systems. The usual approaches to this problem based on discretization of some intrinsic properties such as Lax pairs are very "individual" and not algorithmic. In contrast, Kahan–Hirota–Kimura unconventional discretization is a very straightforward one and can be applied to any Riccati type system, but, generally, it does not guarantee the preserving of integrability property.

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86 Integrable equations, history of

The excellent and detailed accounts on the history of such notions as soliton, higher symmetries, Bäcklund transformation, Painlevé property and so on can be found in [1, 3, 2, 4, 5].

1834 Russel's discovery of great solitary wave of translation[6, 7]

1853 Liouville equation [8]

1855 Liouville definition of integrability [9]

1871 The papers of Boussinesq [10, 11]

1879 Bianchi–Lie–Bäcklund transformation [12, 13, 14, 15, 16]

1882 Darboux transformation [17]

1889 Kowalevski top [18]

1894 sine-Gordon equation [19, 20]

1895 The derivation of KdV equation [21]

1902 Works of Painlevé and Gambier [22, 23]

1910 Tzitzeica equation [24]

1914? Toda lattice [25]

1940 Factorization method [26]

1955 Crum formula [27]

1955 Numerical experiments by Fermi, Pasta, Ulam and Tsingou [28]

1965 Issue of the term "soliton" [29]

1967 Inverse Scattering Transform Method [30]

1967 Darboux lattice [31]

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87 Integrable hierarchy

An equation is called integrable if it possesses an infinite-dimensional algebra of the generalized symmetries. This algebra is called the ***hierarchy*** of the equation under scrutiny.

In a wider sense, one considers as members of the hierarchy also the nonlocal generalized symmetries, the symmetries corresponding to nonisospectral deformations and the discrete symmetries generated by the Bäcklund transformations. This point of view, together with the use of differential/difference substitutions allows to establish useful relations between equations belonging to different classes. All associated equations have the conservation laws and zero curvature representations in common and this allows to apply the unified integration methods to the whole hierarchy.

Example 1. Let us consider the potential KdV equation

$$u_{t_3} = u_{xxx} + 6u_x^2. \tag{1}$$

It admits the higher symmetries

$$u_{t_5} = u_{xxxxx} + 20u_x u_{xxx} + 10u_{xx}^2 + 40u_x^3, \dots$$

generated by the recursion operator $R = D_x^2 + 8u_x - 4D_x^{-1}u_{xx}$. The commutative Lie algebra generated by these flows is what is called the pot-KdV hierarchy.

The BT for equation (1) defines the dressing chain

$$u_{n+1,x} + u_{n,x} + (u_{n+1} - u_n)^2 + a_n = 0.$$

The problem of finding its periodic in n solutions turns out to be equivalent to construction problem of finite-gap solutions of KdV.

Further, the nonlinear superposition principle leads to the discrete KdV equation

$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) = a_n - b_m.$$

The change $v = u_{n+1,m} - u_{n,m}$, $w = u_{n,m+1} - u_{n,m}$ brings to Yang–Baxter map

$$v_2 = -w + \frac{a_1 - a_2}{w - v}, \qquad w_1 = -v + \frac{a_1 - a_2}{w - v},$$

and the restriction onto the even sublattice brings to the discrete Toda-type lattice

$$\sum_n \frac{a_n - a_{n+1}}{u_{n,n+1} - u} = 0.$$

In the wide sense, all these equations can be considered as members of the equation (1) hierarchy.

88 Integrable mapping

Discrete Liouville theorem [1, 2]

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89 Ishimori equation

$$s_t = [s, s_{yy} - s_{xx}] + g_y s_x + g_x s_y, \quad g_{xx} + g_{yy} = 2\langle s, s_y, s_x\rangle, \quad s \in \mathbb{R}^3,\ g \in \mathbb{R}$$

A two-dimensional generalization of Heisenberg equation. It is gauge equivalent to Davey–Stewartson system [2].

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90 Ito system

$$u_t = u_{xxx} + 6uu_x + 2vv_x, \quad v_t = 2(uv)_x$$

➢ Zero curvature representation:

$$\psi_{xx} = \left(\lambda - u - \frac{v^2}{4\lambda}\right)\psi, \quad \psi_t = (4\lambda + 2u)\psi_x - u_x\psi.$$

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91 Jordan algebra

Author: V.V. Sokolov, 04.07.2006

Jordan algebra is a commutative nonassociative algebra with the identities

$$a \circ b = b \circ a, \quad (a \circ b)a^2 = a \circ (b \circ a^2).$$

Any associative algebra A gives rise to the Jordan algebra A^+ with respect to the product $a \circ b = ab + ba$. Jordan algebra which is isomorphic to a subalgebra of some A^+ is called special. There exist Jordan algebras which cannot be obtained in this way, these are called exceptional.

Example 1. Examples of simple Jordan algebras:

1) gl_n^+, that is the algebra of all $n \times n$ matrices with respect to the multiplication $X \circ Y = XY + YX$ with the usual matrix multiplication in r.h.s..

2) The space of n-dimensional vectors with respect to the multiplication

$$a \circ b = \langle a, c\rangle b + \langle b, c\rangle a - \langle a, b\rangle c$$

where $\langle , \rangle$ is a nondegenerate symmetric bilinear form and c is a given constant vector.

Jordan algebras are related with some multifield KdV equations.

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92 Jordan pair

Author: V.V. Sokolov, 04.07.2006

Jordan pair s a direct sum $V = V^+ \oplus V^-$ of vector spaces over a field $\mathbb{F}$ (if the spaces V^+ and V^- coincide then the term ***Jordan triple system*** is used) equipped with a trilinear operation

$$\{\,\}: V^\pm \times V^\mp \times V^\pm \to V^\pm$$

which satisfies the identities

$$\{abc\} = \{cba\}, \tag{1}$$
$$\{ab\{cde\}\} - \{cd\{abe\}\} = \{\{abc\}de\} - \{c\{bad\}e\}. \tag{2}$$

The most important examples of Jordan pairs are:

$$2\{abc\} = \langle a,b\rangle c + \langle c,b\rangle a, \qquad a,b,c \in \mathbb{F}^N, \tag{3}$$
$$\{abc\} = \langle a,b\rangle c + \langle c,b\rangle a - \langle a,c\rangle b, \qquad a,b,c \in \mathbb{F}^N, \tag{4}$$
$$2\{abc\} = abc + cba, \qquad a,c \in \mathrm{Mat}_{M,N}(\mathbb{F}), \quad b \in \mathrm{Mat}_{N,M}(\mathbb{F}). \tag{5}$$

The important role play the operators $L(a,b): V \to V$ defined by formula

$$L(a,b)(c+d) = \{abc\} - \{bad\}, \quad a,c \in V^+, \quad b,d \in V^-.$$

Relation (2) means that $L(a,b) \in \mathrm{Der}(V)$. The differentiations of such type are called ***interior***. Moreover, the identity (2) is equivalent to the commutation rule

$$[L(a,b), L(c,d)] = L(\{abc\},d) - L(c,\{bad\})$$

which implies that all interior differentiations form the Lie subalgebra $\mathrm{Inder}(V) \subseteq \mathrm{Der}(V)$. An example of ***exterior differentiation*** is given by the map

$$\sigma(a+b) = a - b, \quad a \in V^+, \quad b \in V^-.$$

The ***structure Lie algebra*** of the Jordan pair is defined as

$$\mathrm{strl}(V) = V \oplus \mathrm{Der}(V)$$

with the commutator ($a, c \in V^+$, $b, d \in V^-$, $F, G \in \mathrm{Der}(V)$)

$$[a+b+F, c+d+G] = (F(c) - G(a)) + (F(d) - G(b)) + ([F,G] + L(a,d) - L(c,b)).$$

A number of multifield systems is related to the Jordan pairs: analogs of NLS, DNLS and modified Volterra lattice, mKdV, some examples with the rational r.h.s..

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93 Kadomtsev–Petviashvili equation

$$u_t = u_{xxx} - 6uu_x + 6\sigma^2 v_{yy}, \quad 2v_x = u$$

➢ This is probably the most famous 3D equation. It describes long water waves with weak nonlinearity and dispersion; also it can be used as a model for waves in ferromagnetic media or Bose–Einstein condensates. The review of many results can be found in the books [2, 3, 4].

➢ Auxiliary linear problem [5, 6]:

$$\sigma\psi_y = \psi_{xx} - u\psi, \quad \psi_t = 4\psi_{xxx} - \frac{3}{2}u\psi_x - \frac{3}{4}(u_x + 2\sigma v_y)\psi.$$

➢ Bäcklund transformation (x, y-part) [7, 8]:

$$(v_n + v_{n+1})_x = (v_n - v_{n+1})^2 - \sigma g_n, \quad g_{n,x} = (v_n - v_{n+1})_y.$$

➢ Higher symmetry

$$u_{t_4} = u_{xxy} - 4uu_y - 2u_x v_y + w_{yyy}, \quad v_x = u, \quad w_x = v.$$

➢ Hirota bilinear form ($u = 2(\log f)_{xx}$):

$$(D_x D_t + D_x^4 + 3\sigma^2 D_y^2) f \cdot f = 0.$$

➢ N-soliton solution was found in [9]. The soliton solutions were studied also in the papers [10, 11] (analysis of the additional restrictions on the values of the parameters leading to the resonance interaction of solitons), [12, 13, 14, 15] and many others. The properties of the solutions depend essentially on the sign of σ^2. If $\sigma^2 > 0$ then the soliton solution is stable with respect to the 2-dimensional perturbations, while the case $\sigma^2 < 0$ is unstable.

➢ The localized rational solution, or the ***lump***

$$u = 2D_x^2 \log((x + ay + (a^2 - b^2)t)^2 + b^2(y + 2at)^2 + 3b^{-2}),$$

was found in [16, 17]. The formula for the multi-lump solution was also derived there, which demonstrates that the lumps interact without the phase shifts.

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94 Kadomtsev–Petviashvili equation cylindrical

$$u_{xt} = (u_{xx} + 3u^2)_{xx} - \frac{u_x}{2t} + 3\sigma^2 \frac{u_{yy}}{t^2} \tag{1}$$

Alias: Johnson equation [3].

➢ The point equivalence to KP equation [4]:

$$u(x,y,t) = U\Big(x + \frac{y^2 t}{12\sigma^2}, yt, t\Big), \quad U_{XT} = (U_{XX} + 3U^2)_{XX} + 3\sigma^2 U_{YY}.$$

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95 Kadomtsev–Petviashvili equation modified

$$u_t = u_{xxx} - 6u^2 u_x + 6u_x v_y + 3v_{yy}, \quad v_x = u$$

96 Kadomtsev–Petviashvili equation matrix

$$u_t = u_{xxx} - 3(uu_x + u_x u - v_{yy} + v_y u - uv_y), \quad v_x = u, \quad u \in \mathrm{Mat}_n(\mathbb{R})$$

97 Kahan–Hirota–Kimura discretization

Let an ODE system with quadratic r.h.s. be given

$$x' = Q(x,x) + Ax + b, \quad x \in \mathbb{R}^n,$$

where $Q(x,y) = Q(y,x)$ is a rank 3 tensor, A is a matrix and b is a vector. The discretization proposed in the works [1, 2, 3] is given by the formula

$$\frac{x_{n+1} - x_n}{\varepsilon} = Q(x_{n+1}, x_n) - Q(x_{n+1}, x_{n+1}) - Q(x_n, x_n) + A(x_{n+1} + x_n) + 2b,$$

which defines a birational mapping $x_{n+1} = f_\varepsilon(x_n)$ with the property $x_n = f_{-\varepsilon}(x_{n+1})$. In general, this trick does not guarantee the preserving of the Liouville integrability. However, the conjecture exists that if the original ODE system is algebraically completely integrable then this is true for the corresponding discrete version as well.

Examples: Lotka–Volterra system, Euler top

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98 Kaup system

$$u_t = u_{xx} + 2(u+v)u_x, \quad v_t = -v_{xx} + 2(u+v)v_x$$

➣ Bäcklund transformation:

$$u_{n,x} = (u_n + v_{n+1})(u_{n+1} - u_n + \beta_n), \quad v_{n,x} = (u_{n-1} + v_n)(v_n - v_{n-1} - \beta_{n-1})$$

➣ Nonlinear superposition principle

$$\tilde{u}_n = u_n - (\beta_{n+1} - \beta_n)\frac{u_n + v_{n+1}}{u_{n-1} + v_{n+1} - \beta_{n-1}}, \quad \tilde{v}_n = v_n + (\beta_{n+1} - \beta_n)\frac{v_n + u_{n-1}}{u_{n-1} + v_{n+1} - \beta_n}$$

➣ Zero curvature representation

$$U = \begin{pmatrix} \frac{1}{2}(u-v) & (u+\lambda)(v+\lambda) \\ 1 & \frac{1}{2}(v-u) \end{pmatrix}, \quad V = (u+v-2\lambda)U + \begin{pmatrix} \frac{1}{2}(u_x+v_x) & \lambda(u_x - v_x) + u_x v - u v_x \\ 0 & -\frac{1}{2}(u_x+v_x) \end{pmatrix},$$

$$W_n = (u_n + v_{n+1})^{-1/2} \begin{pmatrix} u_n - \lambda & u_n v_{n+1} + (\lambda - \beta_n)(u_n + v_{n+1}) + \lambda^2 \\ 1 & v_{n+1} - \lambda \end{pmatrix}$$

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99 Kaup–Broer system

$$u_t = -u_{xx} + 2uu_x + 2v_x, \quad v_t = v_{xx} + 2(uv)_x$$

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100 Kaup–Kupershmidt equation

$$u_t = u_5 + 5uu_3 + \frac{25}{2}u_1u_2 + 5u^2u_1$$

➢ Lax pair:

$$L = D_x^3 + uD_x + \frac{1}{2}u_1, \quad -A = 9D_x^5 + 15uD_x^3 + \frac{45}{2}u_1D_x^2 + \frac{5}{2}(7u_2 + 2u^2)D_x + 5(u_3 + uu_1).$$

➢ See also: Sawada–Kotera equation

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101 Kaup–Kupershmidt equation, twodimensional

$$u_t = u_5 + 5uu_3 + \frac{25}{2}u_1u_2 + 5u^2u_1 - 5u_{2,y} - 5uu_y - 5wu_1 - 5w_y, \quad u_y = w_x$$

➢ Introduced in [1].

➢ Auxiliary linear problems $\psi_y = L\psi$, $\psi_t = A\psi$,

$$L = D_x^3 + uD_x + \frac{1}{2}u_1, \quad -A = 9D_x^5 + 15uD_x^3 + \frac{45}{2}u_1D_x^2 + \frac{5}{2}(7u_2 + 2u^2 + 2w)D_x + 5(u_3 + 2uu_1 + \frac{1}{2}u_y).$$

➢ See also: 2D Sawada–Kotera equation

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102 Khokhlov–Zabolotskaya equation

$$u_{xt} = u_x u_{xx} + u_{yy} + u_{zz}$$

Reduction $u_z = 0$ corresponds to the dispersionless limit of KP equation.

The Lagrangian: $L = -u_x u_t + \frac{1}{3} u_x^3 + u_y^2 + u_z^2$.

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103 Kirchhoff system

$$u' = [u, H_u] + [v, H_v], \quad v' = [v, H_u], \quad H = \langle u, Au\rangle + \langle v, Bv\rangle + \langle u, Cv\rangle$$
$$u, v \in \mathbb{R}^3, \quad A, B, C \in \mathrm{Mat}_3(\mathbb{R}), \quad A = \mathrm{diag}(a_1, a_2, a_3), \quad B = B^{\mathsf{T}}$$

104 Kolmogorov–Petrovsky–Piskunov equation

$$u_t = u_{xx} + \delta(u-\alpha)(u-\beta)(u-\gamma)$$

Alias: FitzHugh–Nagumo equation

➢ Not integrable. The change $x \to ax$, $t \to a^2 t$, $u \to bu + c$ allows to bring the equation to the form

$$u_t = u_{xx} - u(u-1)(u-\alpha).$$

The rich families of exact solutions were found in [4, 5, 6].

See also: Burgers–Huxley, Fischer equations.

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105 Korteweg–de Vries equation

$$u_t = u_{xxx} + 6uu_x$$

➢ This fundamental equation describes the weakly nonlinear waves in the one dimensional media with weak dispersion. Introduced in [1], it was the first nonlinear equation integrated by use of Inverse Scattering Method [2].

➢ Higher symmetries are defined by the formula $u_{t_{2n+1}} = R^n(u_x) = u_{2n+1} + \dots$ where $R = D_x^2 + 4u + 2u_x D_x^{-1}$ is the recursion operator. For example, next two symmetries are:

$$u_{t_5} = u_5 - 10uu_3 - 20u_1u_2 + 30u^2u_1,$$
$$u_{t_7} = u_7 - 14uu_5 - 42u_1u_4 - 70u_2u_3 + 70u^2u_3 + 280uu_1u_2 + 70u_1^3 - 140u^3u_1.$$

All higher symmetries are local and can be chosen homogeneous with respect to the weight $\mathrm{w}(u_n) = 2 + n$.

Alternatively, higher symmetries $u_{t_{2n+1}} = \mathrm{const}\, D_x(g_n)$ can be computed by use of the generating function $g = 1 + g_1/\lambda + g_2/\lambda^2 + \dots$ accordingly to explicit recurrent relations

$$g_{xxx} + 4(\lambda + u)g_x + 2u_x g = 0 \quad \Rightarrow \quad 2gg_{xx} - g_x^2 + 4(\lambda + u)g^2 = 4\lambda \quad \Rightarrow$$
$$8g_{n+1} = \sum_{j=1}^{n-1} g_{j,x}g_{n-j,x} - 2\sum_{j=0}^{n-1} g_j g_{n-j,xx} - 4\sum_{j=1}^{n} g_j g_{n+1-j} - 4u\sum_{j=0}^{n} g_j g_{n-j}.$$

➢ Zero curvature representation

$$U = \begin{pmatrix} 0 & 1 \\ -u-\lambda & 0 \end{pmatrix}, \quad V_n = (-4)^n \begin{pmatrix} -\frac{1}{2}G_{n,x} & G_n \\ -\frac{1}{2}G_{n,xx} - (u+\lambda)G_n & \frac{1}{2}G_{n,x} \end{pmatrix}, \quad G_n = \lambda^n + \lambda^{n-1}g_1 + \dots + g_n$$

➢ Soliton solutions. A wide class of solutions is given by the formula $u = 2(K(x,x,t))_x$ where $K(x,y,t)$ is a solution of Gelfand–Levitan–Marchenko equation

$$F(x+y,t) + K(x,y,t) + \int_x^\infty F(\xi+y,t)K(x,\xi,t)d\xi = 0, \quad x \le y < \infty$$

with a kernel $F(x,t)$ rapidly decreasing at $x \to -\infty$ and such that $F_t = -8F_{xxx}$. In particular, the n-soliton solutions corresponds to the degenerate kernel $F(x,t) = \sum_{j=1}^{n} \exp(k_j x + 8k_j^3 t + \delta_j)$.

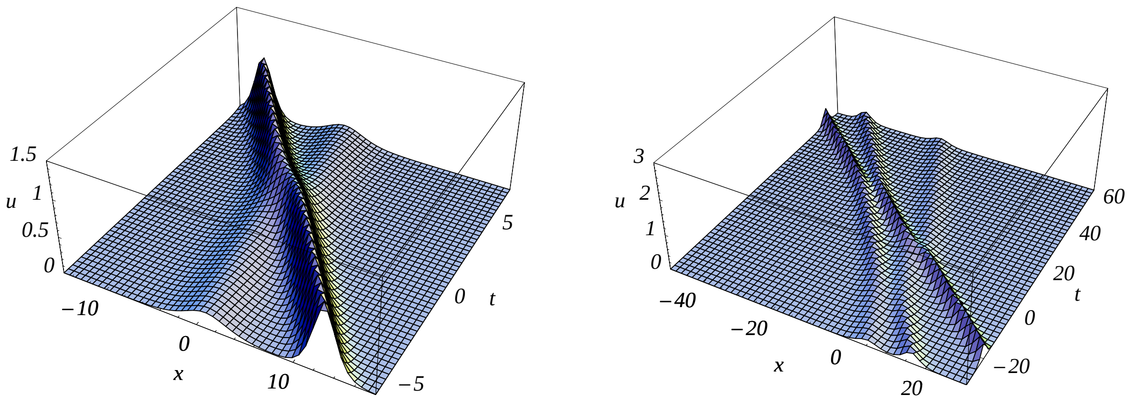
The 1-soliton solution is given by the formula

$$u = \frac{2k^2}{\cosh^2(kx + 4k^3 t + \delta)}.$$

The formula for the N-soliton solution [3] reads

$$u = 2D_x^2 \log W[e^{y_1} + e^{-y_1}, \dots, e^{y_n} - (-1)^n e^{-y_n}], \quad y_j = k_j x + 4k_j^3 t + \delta_j, \quad 0 < k_1 < \dots < k_n$$

where W denotes the Wronskian $W[f_1, \dots, f_n] = \det(D_x^{i-1}(f_j))_{j,i=1}^{n}$.



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106 Korteweg–de Vries equation cylindrical

$$u_t = u_{xxx} + 6uu_x - \frac{u}{2t} \tag{1}$$

➢ The Lax pair [1]:

$$-t\psi_{xx} = \Big(\frac{x}{12} + tu + \lambda\Big)\psi, \quad \psi_t = 4\psi_{xxx} + 6u\psi_x + 3u_x\psi.$$

➢ The point transformation to KdV equation [2]:

$$u(x,t) = \frac{1}{t}U(xt^{-1/2}, -2t^{-1/2}) - \frac{x}{12t}, \quad U_T = U_{XXX} + 6UU_X.$$

See also [4, 5]. Another equivalent form is [3]

$$u_t = u_{xxx} + 6t^{-1/2}uu_x.$$

➢ Recursion operator for the latter form is [6]

$$L = tD_x^2 + 4t^{1/2}u + \frac{1}{3}x + (2t^{1/2}u_x + \frac{1}{6})D_x^{-1}.$$

➢ Multifield generalizations were studied in [7].

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107 Korteweg–de Vries equation Jordan

$$u_t = u_{xxx} + u \circ u_x, \quad u \in J$$

where J is a Jordan algebra. The particular cases are vector and matrix KdV equations.

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[1] S.I. Svinolupov. Jordan algebras and generalized KdV equations. *Theor. Math. Phys.* ***87:3*** (1991) 611–620.

108 Korteweg–de Vries equation matrix

$$u_t = u_{xxx} + 3uu_x + 3u_x u, \quad u \in Mat_n$$

109 Korteweg–de Vries equation modified

$$u_t = u_{xxx} \pm 6u^2 u_x$$

110 Korteweg–de Vries equation modified Jordan

$$u_t = u_{xxx} + \{u, u, u_x\}, \quad u \in J$$

where J Jordan triple systems.

The particular cases are the following vector and matrix analogs of mKdV equation:

$$u_t = u_{xxx} + \langle u, u \rangle u_x, \quad u \in \mathbb{R}^N,$$
$$u_t = u_{xxx} + \langle u, u \rangle u_x + \langle u, u_x \rangle u, \quad u \in \mathbb{R}^N,$$
$$u_t = u_{xxx} + u^2 u_x + u_x u^2, \quad u \in \mathrm{Mat}_N .$$

111 Korteweg–de Vries equation modified matrix–1

$$u_t = u_{xxx} + 3u^2 u_x + 3u_x u^2, \quad u \in Mat_n \qquad (1)$$

112 Korteweg–de Vries equation modified matrix–2

$$u_t = u_{xxx} + 3[u, u_{xx}] + 6uu_xu, \quad u \in Mat_n \tag{1}$$

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[1] F.A. Khalilov, E.Ya. Khruslov. Matrix generalization of the modified Korteweg-de Vries equation. *Inverse Problems* ***6:2*** (1990) 193–204.

[2] Q.P. Liu, C. Athorne. Comment on ‘Matrix generalization of the modified Korteweg-de Vries equation’. *Inverse Problems* ***7:5*** (1991) 783–785.

113 Korteweg–de Vries equation potential

$$u_t = u_{xxx} + 6u_x^2$$

- Substitution $v = 2u_x$ brings to KdV equation $v_t = v_{xxx} + 6vv_x$.
- Recursion operator: $R = D_x^2 + 8u_x - 4D_x^{-1}u_{xx}$.
- Higher symmetry:

$$u_t = u_{xxxxx} + 20u_x u_{xxx} + 10u_{xx}^2 + 40u_x^3$$

114 Korteweg–de Vries equation with Schwarzian

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{2u_x} + au_x$$

Most degenerate case of Krichever–Novikov equation.

115 Korteweg–de Vries equation spherical

$$u_t = u_{xxx} + 6uu_x + \frac{u}{t}$$

Not integrable, in contrast to the cylindrical KdV equation.

116 Korteweg–de Vries equation, super-

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x + 3vv_x, \quad v_t = v_{xxx} + \frac{3}{2}uv_x + \frac{3}{4}u_x v$$

References

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117 Korteweg–de Vries equation modified vectorial

$$u_t = u_{xxx} + \langle c, u \rangle u_x + \langle c, u_x \rangle u - \langle u, u_x \rangle c, \qquad u, c \in \mathbb{R}^d, \quad c = \text{const}\,.$$

118 Korteweg–de Vries-type equations, classification

Authors: A.G. Meshkov, V.V. Sokolov, 2009

1. The list of integrable equations

KdV-type equations are integrable third order evolutionary equations with constant separant:

$$u_t = u_3 + F(u, u_1, u_2), \tag{1}$$

Their exhaustive classification was obtained by Svinolupov and Sokolov [1, 2] (more precisely, a bit more general problem with F explicitly depending on x was solved in these papers, however, it turned out that it did not lead to essentially new answers). The proof of the following theorem can be transformed to an integrability test which can be applied to a given equation of the form (1). Moreover, if equation happens to be integrable then the change of variables relating it to one of the equation in the list is found constructively.

Theorem 1. *Any nonlinear integrable equation (1) is point equivalent to an equation from the following list:*

$$u_t = u_3 + uu_1, \tag{K$_1$}$$

$$u_t = u_3 + u^2u_1, \tag{K$_2$}$$

$$u_t = u_3 + u_1^2, \tag{K$_3$}$$

$$u_t = u_3 - \frac{1}{2}u_1^3 + (c_1e^{2u} + c_2e^{-2u})u_1, \tag{K$_4$}$$

$$u_t = u_3 - \frac{3u_1u_2^2}{2(u_1^2+1)} + c_1(u_1^2+1)^{3/2} + c_2u_1^3, \tag{K$_5$}$$

$$u_t = u_3 - \frac{3u_1u_2^2}{2(u_1^2+1)} - \frac{3}{2}P(u)u_1(u_1^2+1), \tag{K$_6$}$$

$$u_t = u_3 - \frac{3(u_2^2-1)}{2u_1} - \frac{3}{2}P(u)u_1^3, \tag{K$_7$}$$

$$u_t = u_3 - \frac{3u_2^2}{2u_1}, \tag{K$_8$}$$

$$u_t = u_3 - \frac{3u_2^2}{4u_1} + c_1 u_1^{3/2} + c_2 u_1^2, \quad (c_1, c_2) \neq 0, \tag{K$_9$}$$

$$u_t = u_3 - \frac{3u_2^2}{4u_1} + cu, \tag{K$_{10}$}$$

$$\begin{aligned} u_t = u_3 &- \frac{3u_2^2}{4(u_1+1)} - 3u_1(u_1+1) + 3u_2\sqrt{u_1+1} \\ &- 6u_1(u_1+1)^{3/2} + 3u_1(u_1+2)(u_1+1), \end{aligned} \tag{K$_{11}$}$$

$$\begin{aligned} u_t = u_3 &- \frac{3u_2^2}{4(u_1+1)} - 3\frac{u_2(u_1+1)\cosh u}{\sinh u} + 3\frac{u_2\sqrt{u_1+1}}{\sinh u} \\ &- 6\frac{u_1(u_1+1)^{3/2}\cosh u}{\sinh^2 u} + 3\frac{u_1(u_1+2)(u_1+1)}{\sinh^2 u} + u_1^2(u_1+3), \end{aligned} \tag{K$_{12}$}$$

$$u_t = u_3 + 3u^2u_2 + 3u^4u_1 + 9uu_1^2, \tag{K$_{13}$}$$

$$u_t = u_3 + 3uu_2 + 3u^2u_1 + 3u_1^2, \tag{K$_{14}$}$$

where $(P')^2 = 4P^3 - g_2P - g_3$ *and* k, c, c_1, c_2, g_2, g_3 *are arbitrary constants.*

Remark 2. Equations (K$_1$)–(K$_9$) are S-integrable, and (K$_{10}$)–(K$_{14}$) are C-integrable.

Remark 3. Equations

$$u_t = u_3 + \frac{3((Q-u_1^2)_x)^2}{8u_1(Q-u_1^2)} - \frac{1}{2}Q''u_1 \qquad \text{and} \qquad u_t = u_3 - \frac{3(u_{xx}^2+Q)}{2u_1}$$

where $Q = c_4u^4 + c_3u^3 + c_2u^2 + c_1u + c_0$ is an arbitrary polynomial of 4-th degree are another canonical forms of equations (K$_6$) and (K$_7$) respectively. Namely, let $Q \neq 0$ then the change $u = f(v)$, where $(f')^2 = -Q(f)$, brings these equations to equations (K$_6$) and (K$_7$) for the variable v.

Remark 4. The point transformations used for bringing KdV-type equations to one of the listed above are rather simple. The whole class of equations (1) admits the following point transformations:

$$\text{(conformal changes)} \qquad \tilde{u} = \phi(u), \tag{2}$$

$$\text{(Galilean boost)} \qquad \tilde{x} = x + ct, \qquad F \to F - cu_1, \tag{3}$$

$$\text{(scaling)} \qquad \tilde{x} = ax, \quad \tilde{t} = a^3 t, \qquad F(u, u_1, u_2) \to a^{-3} F(u, au_1, a^2 u_2). \tag{4}$$

Moreover, some special subclasses of equations admits additional point transforms. If function F does not depend on u then the transformation

$$\tilde{u} = u + c_1 x + c_2 t, \qquad F(u_1, u_2) \to F(u_1 - c_1, u_2) + c_2 \tag{5}$$

as admissible, and if the function F is homogeneous of the weight 1: $F(\lambda u, \lambda u_1, \lambda u_2) = \lambda F(u, u_1, u_2)$, then an admissible transformation is

$$\tilde{u} = u \exp(at + bx), \qquad F \to F + au, \quad u_n \to (\partial_x - b)^n u. \tag{6}$$

The scheme of the proof presented below gives simultaneously an algorithm of reducing an integrable equation to one of the standard forms (K_1)–(K_{14}).

2. Integrability conditions

The definition of integrability for equations of KdV type requires the existence of higher infinitesimal symmetries and/or higher conservation laws. The classification of equations with such properties is based on the symmetry approach which is especially effective in the case of evolutionary equations with one spatial variable.

The invariant description of all integrable equations (1) is given by the following statement which means that an integrable equation must possess the local conservation laws $(\rho_n)_t = (\sigma_n)_x$, $n = 0, 1, \dots$ with the densities and fluxes recursively defined through the r.h.s. of the equation. Recall, that these conservation laws are called canonical. It should be explained that although the symmetry approach gives only necessary integrability conditions, actually just a few of these conditions are enough for the complete classification (four ones in the case under consideration) and after the answers are found, the integrability of each equation is proven individually.

Theorem 5. *Equation (1) possesses an infinite series of higher symmetries if and only if the following integrability conditions are fulfilled:*

$$D_t(F_{u_2}) = D_x(\sigma_0), \tag{7}$$

$$D_t(3F_{u_1} - F_{u_2}^2) = D_x(\sigma_1), \tag{8}$$

$$D_t(9\sigma_0 + 2F_{u_2}^3 - 9F_{u_2}F_{u_1} + 27F_u) = D_x(\sigma_2), \tag{9}$$

$$D_t(\sigma_1) = D_x(\sigma_3). \tag{10}$$

where $F_{u_i} = \partial_{u_i}(F)$, D_x is the operator of total derivative with respect to x and D_t is evolutionary derivative in virtue of equation (1).

Concerning the proof of this theorem, we mention that the integrability conditions follow from the existence of the formal symmetry. There is also another method [3, 4] of the computation of canonical densities through the logarithmic derivative of the formal eigenfunction for the operator of linearization of equation (1). It brings to the recurrent formula

$$\rho_{n+2} = \frac{1}{3}\left[\sigma_n - \delta_{n,0}F_u - F_{u_1}\rho_n - F_{u_2}\left(D_x(\rho_n) + 2\rho_{n+1} + \sum_{s=0}^{n}\rho_s\rho_{n-s}\right)\right] - \sum_{s=0}^{n+1}\rho_s\rho_{n+1-s}$$

$$-\frac{1}{3}\sum_{0\le s+k\le n}\rho_s\rho_k\rho_{n-s-k}-D_x\left[\rho_{n+1}+\frac{1}{2}\sum_{s=0}^{n}\rho_s\rho_{n-s}+\frac{1}{3}D_x(\rho_n)\right],\qquad n\ge 0$$

where $\delta_{i,j}$ is Kronecker delta and the initial data are

$$\rho_0=-\frac{1}{3}F_{u_2},\qquad \rho_1=\frac{1}{9}F_{u_2}^2-\frac{1}{3}F_{u_1}+\frac{1}{3}D_x(F_{u_2}).$$

It is easy to check that the first four conditions from this sequence are equivalent to the conditions (7)–(10).

The effective use of the canonical conservation laws for the classification is based on the preliminary study of the possible structure of the densities of the local conservation laws for equations of the form (1).

Lemma 6. *let $\rho(u,u_1,u_2)$ be a conserved density for equation (1). Then*

$$\rho_{u_2u_2u_2}=0,\qquad \rho_{u_2u_2u_1}+\rho_{u_*u_*u_*}=\frac{2}{3}F_{u_2}\rho_{u_2u_2}. \tag{11}$$

The following algorithm is used in the proof in order to check, if the given function $S(u,u_1,\dots,u_n)$ is the total derivative in x (that is, belongs to $\operatorname{Im}D_x$) or not. First, S have to be linear in the leading derivative u_n. If this is true then, as one can easily see, a total derivative can be subtracted from S in such a way that the order of the result will be less than n. Repeating of this procedure we come either to an expression which is not linear in the leading derivative or to zero.

An alternative method is based on the well-known property

$$S\in\mathbb{R}\oplus\operatorname{Im}D_x\quad\Leftrightarrow\quad\frac{\delta S}{\delta u}=0,\qquad \frac{\delta}{\delta u}:=\sum_{k=0}^{\infty}(-D_x)^k\partial_{u_k}$$

of variational derivative. Although it is more transparent theoretically, the previous method is much more effective for computation.

Let us show, how the formulae (11) are used in the classification of equations (1).

Lemma 7. *Let equation (1) satisfies the first integrability condition (7). Then function F is quadratic in u_2.*

Proof. Accordingly to the first equation (11),

$$F_{u_2} = f_1 u_2^2 + f_2 u_2 + f_3.$$

Substitute this expression into the second equation (11), this gives

$$f_{1,u} u_1 + f_{1,u_1} u_2 = \frac{2}{3} f_1 (f_1 u_2^2 + f_2 u_2 + f_3).$$

Since f_i does not depend on u_2, hence balancing of the coefficients at u_2^2 yields $f_1 = 0$. Integration of equation $F_2 = f_2 u_2 + f_3$ proves the lemma. ■

The analogous computations related with the next integrability conditions allow to determine the possible dependence of the function F on u_1 and finally bring to the complete list of integrable equations (1), up to the point changes given above. The brief sketch of these reasonings is given in the next section.

3. The classification scheme

Accordingly to the Lemma 7 the equation is of the form

$$u_t = u_3 + A_2(u_1, u) u_2^2 + A_1(u_1, u) u_2 + A_0(u_1, u), \tag{12}$$

moreover this form is invariant under any admissible transformation (2)–(6). It is easy to obtain from the integrability condition (8) that

$$9A_{2,u_1 u_1} - 36 A_2 A_{2,u_1} + 16 A_2^3 = 0,$$

whence

$$A_2 = -\frac{3B_{u_1}}{4B}, \qquad \text{where} \quad B_{u_1 u_1 u_1} = 0.$$

Case 1. Let the degree of the polynomial B is equal 2: $B = u_1^2 + B_1(u) u_1 + B_0(u)$ then the condition (8) implies

$$A_1 = -\frac{3B_u}{2B} u_1.$$

The condition (7) is fulfilled for any such equation. This means that the function σ_0 is known and we can use the condition (9), if we wish. The condition (8) gives that $B_1 B_0' = 2B_1' B_0$. It is easy to check that this relation implies that a suitable point transformation $u \to \phi(u)$ allows to make the polynomial B not depending on u: $B = u_1^2 + \beta_1 u_1 + \beta_0$. Clearly, then $A_1 = 0$. Then, we find for the function A_0 that

$$2BA_{0,u_1u_1u_1u_1} + 3B'A_{0,u_1u_1u_1} - 3B''A_{0,u_1u_1} = 0.$$

1.1. In the case of the distinct zeroes of B the solution of this equation is

$$A_0 = k_1(u)B^{3/2} + k_2(u)(2u_1^3 + 3\beta_1 u_1^2) + k_3(u)u_1 + k_4(u).$$

Then it follows from the condition (8) that if the coefficient $k_2(u)$ is constant then all other coefficients are constant as well and we come (up to the admissible transformations) to equation (K_5). In the case $k_2' \neq 0$ we obtain equation (K_6). Thus, two first integrability conditions are enough if the zeroes are distinct.

1.2. In the case of double zero $B = (u_1 + z)^2$ we have

$$A_0 = \frac{k_1(u)}{u_1 + z} + k_2(u)(u_1^3 + 3zu_1^2) + k_3(u)u_1 + k_4(u).$$

If $z \neq 0$ then all coefficients turn out to be constant and we can use transformations (2)–(5) in order to bring equations to the form (K_8) or to the form (K_7) with $P(u) = const$.

In the case $z = 0$ we still have not used the transformations $u \to \phi(u)$ for bringing the equation to the canonical form. This change allows to make the function $k_1(u) \neq 0$ constant, and if $k_1(u) = 0$ then it is possible to set $k_2 = 0$.

If $k_1 = 0$ then $k_2 = 0$ and, in virtue of condition (8), $k_3' = 0$. Further, condition (10) implies $k_4' = 0$ and we come back to the case $z \neq 0$ which is already studied.

If $k_1 \neq 0$ then, up to the transformation (4), $k_1 = 3/2$. Then we obtain by use of condition (8) that $k_3' = k_4' = 0$, $k_4 k_2' = 0$. The case $k_2' = 0$ brings to the same result as the case $z \neq 0$ above. In the case $k_2' \neq 0$ we have $k_4 = 0$, and the constant k_3 is eliminated by Galilean boost. We come to equation (K_7) after determining the function k_2 by use of condition (10).

Case 2. Let the polynomial B is of the first degree: $B = u_1 + B_0(u)$. Then the transformation $u \to \phi(u)$ can be used in order to make B_0 constant: $B = u_1 + \beta_0$. Then the first integrability condition (7) gives

$$A_1 = q_1(u) + q_2(u)u_1 + q_3(u)\sqrt{u_1 + \beta_0}, \quad 3q_1' = q_3^2, \quad 3q_3' = q_2 q_3, \quad q_3(q_1 - \beta_0 q_2) = 0. \tag{13}$$

2.1. Consider the case $q_3 = 0$ first. We have $A_1 = c_0 + q_2(u)u_1$ and the condition (7) is fulfilled.

2.1.a. If $\beta_0 = 0$ then the use of suitable change $u \to \phi(u)$ allows to set $q_2(u) = 0$. Then the conditions (8) and (9) proves that $c_0 = 0$ and

$$A_0 = c_1 u_1^{3/2} + c_2 u_1^2 + c_3 u_1 + c_4 u + c_5,$$

moreover $c_1 c_4 = 0$. The condition (10) gives only one additional constraint $c_2 c_4 = 0$.

If $c_4 = 0$ then we come to equation (K_9) and otherwise to equation (K_{10}), up to transformations (2)–(6).

2.1.b. If $\beta_0 \neq 0$ then it is easy to find, by use of (8) and (9), that the r.h.s. of equation does not depend on u. Then the transformation $u \to u - \beta_0 x$ allows to set the constant β_0 to zero and to reduce this case to the previous one.

2.2. If $q_3 \neq 0$ then equations (13) imply $\beta_0 \neq 0$. The use of the scaling and shift of u allows to obtain from (13): $\beta_0 = 1$, $q_1 = q_2 = -3\coth u$, $q_3 = 3\sinh^{-1} u$ or $\beta_0 = 1$, $q_1 = q_2 = -3$, $q_3 = 0$. Then the conditions (7) and (8) allow to determine A_0 and we come to equations (K_{11}), (K_{12}).

Case 3. Let B is of zero degree, then $A_2 = 0$. The integrability condition (7) gives $A_1 = q(u) + p(u)u_1$. The use of the transformation $u \to \phi(u)$ allows to set $p(u) = 0$. The condition (8) implies

$$A_0 = p_1(u) + p_2(u)u_1 + p_3(u)u_1^2 + cu_1^3, \quad qc = qp_3' = 0.$$

The further analysis depends essentially on the function $A_1 = q(u)$.

3.1. If $q = 0$ then $\rho_0 = 0$ and condition (8) gives, in particular, relations $p_2' p_1 = p_2' p_3 = 0$. If $p_2' \neq 0$ then condition (7) leads either to equation (K_4) (at $c \neq 0$) or to (K_1) or to (K_2). If $p_2' = 0$ then we obtain, using additionally the conditions (9) and (10), the equation is either linear or all p_i are constant. Further, the transformations (2)–(6) bring either to (K_3) or to (K_4) at $c_1 = c_2 = 0$.

3.2. If $q \neq 0$ then we obtain, taking into account the relations $c = p_3' = 0$ and with the use of condition (7) that $q = c_1 + c_2 u + c_3 u^2$, $q' p_1 = 0$. The further analysis is not difficult and the use of all four integrability conditions proves that equation coincides with one of the equations (K_{12}), (K_{13}), up to the transformations (2)–(6).

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119 Korteweg–de Vries-type equations, substitutions

Author: V.E. Adler, 2007

Notations:

➢ in the substitution marked $A \to B$ the tilded variables correspond to equation B;

➢ therefore, in the sequence $A \to B \to C$ the variables corresponding to B go with tilde in the first substitution and without tilde in the second one;

➢ for short, the dummy n in the lattice indices $\ldots, n-1, n, n+1, \ldots$ is omitted;

➢ the letters α, β, γ are reserved for the parameters of Bäcklund transformations. They are always assumed to be dependent on n.

Any integrable KdV-type equation $u_t = u_{xxx} + f(u_{xx}, u_x, u)$ is related by a differential substitution either to the Krichever–Novikov equation or to the KdV equation or to the linear equation [1]. While the first case may be considered generic, the second one is most complicated from the point of view of diversity of contained equations. Famous ***Miura transformation*** [2] relates KdV with the modified KdV equation. Another very important persons in the zoo are so-called exponential and elliptical Calogero–Degasperis equations (exp-CD and $\wp$-CD) related to mKdV by further Miura-type substitutions.

The full picture is presented below. It contains also substitutions for the corresponding dressing chains, since discrete and continuous substitutions are closely related [3, 4]. It should be noted that mKdV equation admits two essentially different Bäcklund transformations. First one is inherited from BT for KdV; it corresponds to zero curvature representation related to the Schrödinger operator and to the minus sign of nonlinear term in mKdV. Second one corresponds to the Dirac operator, and it handles nonlinearity of any sign. Analogously, exp-CD admits three different BT, two inherited from KdV and mKdV and one for its own.

1. Equations related to KdV

$$\begin{array}{ccccc} 1 & \rightarrow & 2 & \leftarrow & 3 \\ \downarrow & \searrow & \downarrow & \swarrow & \\ 4 & \rightarrow & 5 & & \\ & \searrow & \downarrow & & \\ & & 6 & & \\ & & \downarrow & & \\ & & 7 & & \end{array}$$

$$u_t = u_{xxx} - 6u_x^2, \quad u_{1,x} + u_x = (u_1 - u)^2 + \beta \qquad \text{pot-KdV (1)}$$

$$u_t = u_{xxx} - 6uu_x, \quad u_{1,x} + u_x = (u_1 - u)\sqrt{2(u_1 + u) - 4\beta} \qquad \text{KdV (2)}$$

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{4u_x} - 4u_x^{3/2} \qquad (3)$$

$$(y + 4\beta)\sqrt{2y - 4\beta} = 3(u_1 - u), \quad y = \sqrt{u_{1,x}} + \sqrt{u_x}$$

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{4(u_x - \beta)} - 3u_x^2, \quad \sqrt{u_{1,x} - \beta_1} + \sqrt{u_x - \beta} = u_1 - u \qquad (4)$$

$$u_t = u_{xxx} - 6(u^2 + \beta)u_x \qquad \text{mKdV (5)}$$

$$u_{1,x} + u_x = u_1^2 - u^2 + \alpha, \quad \alpha = \beta_1 - \beta$$

$$u_t = u_{xxx} - 3u_x\left(\frac{u_{xx}}{u} - \frac{u_x^2 - \alpha^2}{2u^2} + \frac{u^2}{2} - \beta - \beta_{-1}\right) \qquad \text{exp-CD (6)}$$

$$(u_1u)_x = u_1u(u_1 - u) + \alpha_1 u + \alpha u_1$$

$$u_t = u_{xxx} - \frac{3u_x(u_{xx} + 2r'(u))^2}{2(u_x^2 + 4r(u))} + 6(2u - \beta_1 + \beta - \beta_{-1})u_x \qquad \wp\text{-CD (7)}$$

$$(R_1 + u_{1,x})(R + u_x) = 4u_1(u_1 + \alpha_1)(u + \alpha)$$

$$r(u) = u(u + \alpha)(u - \alpha_1), \quad R^2 = u_x^2 + 4r(u)$$

$1 \to 2: \ \tilde{u} = 2u_x$

$1 \to 4: \ \tilde{u} = u_1 + u$

$1 \to 5: \ \tilde{u} = u_1 - u$

$2 \to 5: \ 2(\tilde{u}^2 + \beta) = u_1 + u$

$u = \tilde{u}^2 - \tilde{u}_x + \beta$

$2\tilde{u}_x = u_1 - u$

$3 \to 2: \ \tilde{u}^2 = u_x$

$3 \to 5: \ \tilde{u}^3 + 3\beta\tilde{u} = \frac{3}{4}(u_1 - u$

$4 \to 5: \ \tilde{u}^2 + \beta = u_x$

$4 \to 6: \ \tilde{u}_1 = u_1 - u$

$5 \to 6: \ \tilde{u}_1 = u_1 + u$

$2u = \tilde{u} + \frac{\tilde{u}_x - \alpha}{\tilde{u}}$

$6 \to 7: \ \tilde{u} = u_1 u$

$2u = \frac{R(\tilde{u}) - \tilde{u}_x}{\tilde{u} - \alpha_1}$

2. Equations related to mKdV

$$\begin{array}{ccccccc} 1 & \rightarrow & 2 & \rightarrow & 3 & \leftarrow & 4 \\ & \searrow & \downarrow & \searrow & \downarrow & \swarrow & \\ & & 5 & \rightarrow & 6 & & \\ & & & \searrow & \downarrow & & \\ & & & & 7 & & \end{array}$$

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{2u_x}, \quad u_{1,x}u_x = (u_1 - \beta u)^2 \qquad \text{Schwarz-KdV } (1)$$

$$u_t = u_{xxx} - 2u_x^3, \quad u_{1,x} + u_x = e^{u_1 - u} - \beta e^{u - u_1} \qquad (2)$$

$$u_t = u_{xxx} - 6u^2 u_x, \quad u_{1,x} + u_x = (u_1 - u)\sqrt{(u_1 + u)^2 + 4\beta} \qquad \text{mKdV } (3)$$

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{4u_x} - 3u_x^2, \quad (\sqrt{u_{1,x}} + \sqrt{u_x})^2 = (u_1 - u)^2 - 4\beta \qquad (4)$$

$$u_t = u_{xxx} - \frac{3u_x u_{xx}^2}{2(u_x^2 + \beta)} - 2u_x^3 \qquad (5)$$
$$(u_{1,x} + \sqrt{u_{1,x}^2 + \beta_1})(u_x + \sqrt{u_x^2 + \beta}) = e^{2(u_1 - u)}$$

$$u_t = u_{xxx} - \frac{3u_x u_{xx}}{u} + \frac{3u_x^3}{2u^2} - \frac{3}{2}\left(u - \frac{\beta}{u}\right)^2 u_x \qquad \text{exp-CD } (6)$$
$$(u_1 u)_x = u_1 u(u_1 - u) - \beta_1 u + \beta u_1$$

$$u_t = u_{xxx} - \frac{3u_x(u_{xx} + 2r'(u))^2}{2(u_x^2 + 4r(u))} + 6(2u + \beta + \beta_{-1})u_x \qquad \wp\text{-CD } (7)$$
$$(R_1 + u_{1,x})(R + u_x) = 4u_1(u_1 + \beta)(u + \beta_{-1})$$
$$r(u) = u(u + \beta)(u + \beta_{-1}), \quad R^2 = u_x^2 + 4r(u)$$

$1 \to 2: \ e^{2\tilde u} = u_x$

$1 \to 5: \ 2\tilde u = \log(u_1 - \beta u)$

$2 \to 3: \ \tilde u = u_x$

$2 \to 5: \ 2\tilde u = u_1 + u$

$\quad 2u = 2\tilde u - \log(\tilde u_x + \sqrt{\tilde u_x^2 + \beta})$

$2 \to 6: \ \tilde u = e^{u_1 - u}$

$3 \to 6: \ \tilde u - \beta/\tilde u = u_1 + u$

$\quad 2u = \tilde u - (\tilde u_x + \beta)/\tilde u$

$4 \to 3: \ \tilde u^2 = u_x$

$4 \to 6: \ \tilde u + \beta/\tilde u = u_1 - u$

$5 \to 6: \ \tilde u - \beta/\tilde u = 2u_x$

$5 \to 7: \ \tilde u_1 = e^{2(u_1 - u)}$

$6 \to 7: \ \tilde u_1 = u_1 u$

$$2u = \frac{R(\tilde u) + \tilde u_x}{\tilde u + \beta_{-1}}$$

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120 Korteweg–de Vries-type equations, 5-th order

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The integrable equations of the 5-th order had been classified only in the constant separant case

$$u_t = u_5 + F(u_4, u_3, u_2, u_1, u). \tag{1}$$

The complete list can be found in [1]. Here we present only the most important equations.

The equations (3)–(10) in the list below appeared in [2, 3, 4, 5, 6]. The rest of the list appeared in [1] for the first time (note that there was a misprint in the form of eq. (13)).

The classification is based on the analysis of the necessary integrability conditions. At the first steps, it can be proved that any equation (1) possessing higher conservation laws is of the form

$$\begin{aligned} u_t = u_5 &+ A_1 u_2 u_4 + A_2 u_4 + A_3 u_3^2 + A_4 u_2^2 u_3 + A_5 u_2 u_3 + A_6 u_3 \\ &+ A_7 u_2^4 + A_8 u_2^3 + A_9 u_2^2 + A_{10} u_2 + A_{11}, \quad A_i = A_i(u, u_1). \end{aligned}$$

The further analysis splits in many subcases and may be very lengthy in the most degenerate ones. For example, the equation [7] $u_t = u_5 + uu_1$ is not integrable but satisfies 10 first integrability conditions.

1. Polynomial and exponential equations

KdV eq. (118.K$_1$) symmetry $\quad u_t = u_5 + 10uu_3 + 20u_1u_2 + 30u^2u_1, \quad (2)$

Sawada–Kotera eq. $\quad u_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1, \quad (3)$

Kaup–Kupershmidt eq. $\quad u_t = u_5 + 5uu_3 + \frac{25}{2}u_1u_2 + 5u^2u_1, \quad (4)$

$$u_t = u_5 + 5u_1u_3 + \frac{5}{3}u_1^3, \tag{5}$$

$$u_t = u_5 + 5u_1u_3 + \frac{15}{4}u_2^2 + \frac{5}{3}u_1^3, \tag{6}$$

[2, 3, 4]

$$u_t = u_5 + 5(u_1 - u^2)u_3 + 5u_2^2 - 20uu_1u_2 - 5u_1^3 + 5u^4u_1, \tag{7}$$

$$u_t = u_5 + 5(u_2 - u_1^2)u_3 - 5u_1u_2^2 + u_1^5, \tag{8}$$

[5]

$$\begin{aligned} u_t = u_5 &+ 5(u_2 - u_1^2 + \lambda_1 e^{2u} - \lambda_2^2 e^{-4u})u_3 - 5u_1u_2^2 \\ &+ 15(\lambda_1 e^{2u} + 4\lambda_2^2 e^{-4u})u_1u_2 + u_1^5 - 90\lambda_2^2 e^{-4u}u_1^3 \\ &+ 5(\lambda_1 e^{2u} - \lambda_2^2 e^{-4u})^2 u_1, \end{aligned} \tag{9}$$

[6]

$$\begin{aligned} u_t = u_5 &+ 5(u_2 - u_1^2 - \lambda_1^2 e^{2u} + \lambda_2 e^{-u})u_3 - 5u_1u_2^2 \\ &- 15\lambda_1^2 e^{2u}u_1u_2 + u_1^5 + 5(-\lambda_1^2 e^{2u} + \lambda_2 e^{-u})^2 u_1. \end{aligned} \tag{10}$$

Substitutions:

$(5) \to (3): \quad u_1 = \tilde{u};$
$(6) \to (4): \quad u_1 = \tilde{u};$
$(7) \to (3): \quad -u_1 - u^2 = \tilde{u};$
$(7) \to (4): \quad 2u_1 - u^2 = \tilde{u};$
$(8) \to (7): \quad u_1 = \tilde{u};$

$(9) \to (4): \quad 2u_2 - u_1^2 \pm 6\lambda_2 e^{-2u}u_1 + \lambda_1 e^{2u} - \lambda_2^2 e^{-4u} = \tilde{u};$
$(10) \to (3): \quad -u_2 - u_1^2 \pm 3\lambda_1 e^u u_1 - \lambda_1^2 e^{2u} + \lambda_2 e^{-u} = \tilde{u};$
$(9)|_{\lambda_1=-\lambda^2,\lambda_2=0} = (10)|_{\lambda_1=\lambda,\lambda_2=0};$
$(9)|_{\lambda_1=\lambda_2=0} = (10)|_{\lambda_1=\lambda_2=0} = (8).$

2. Rational and elliptic equations

$$u_t = u_5 - \frac{5}{u_1}u_2u_4 + \frac{5}{u_1^2}u_2^2u_3 + 5\Big(\frac{\mu_1}{u_1} + \mu_2u_1^2\Big)u_3 - 5\Big(\frac{\mu_1}{u_1^2} + \mu_2u_1\Big)u_2^2 - 5\frac{\mu_1^2}{u_1} + 5\mu_1\mu_2u_1^2 + \mu_2^2u_1^5, \tag{11}$$

$$\begin{aligned} u_t = u_5 &- \frac{5}{u_1}u_2u_4 - \frac{15}{4u_1}u_3^2 + \frac{65}{4u_1^2}u_2^2u_3 + 5\Big(\frac{\mu_1}{u_1} + \mu_2u_1^2\Big)u_3 \\ &- \frac{135}{16u_1^3}u_2^4 - 5\Big(\frac{7\mu_1}{4u_1^2} - \frac{\mu_2u_1}{2}\Big)u_2^2 - 5\frac{\mu_1^2}{u_1} + 5\mu_1\mu_2u_1^2 + \mu_2^2u_1^5, \end{aligned} \tag{12}$$

$$u_t = u_5 - \frac{5}{2u_1}u_2u_4 - \frac{5}{4u_1}u_3^2 + \frac{5}{u_1^2}u_2^2u_3 + \frac{5u_1^{-1/2}}{2}u_2u_3 + 5(u_1 - 2\mu u_1^{1/2} + \mu^2)u_3 - \frac{35}{16u_1^3}u_2^4 - \frac{5u_1^{-3/2}}{3}u_2^3 + 5\Big(\mu u_1^{-1/2} - \frac{3\mu^2}{4u_1} - \frac{1}{4}\Big)u_2^2 + \frac{5u_1^3}{3} - 8\mu u_1^{5/2} + 15\mu^2u_1^2 - \frac{40\mu^3u_1^{3/2}}{3} + 5\mu^4u_1, \tag{13}$$

$$u_t = u_5 - \frac{15(R^5+2R^2)}{2(R^3-1)^2}u_2u_4 - \frac{45R^2}{4(R^3-1)^2}u_3^2 + \frac{45(R^{10}+22R^7+13R^4)}{4(R^3-1)^4}u_2^2u_3 + 5\mu R^2u_3 - \frac{3645(2R^{12}+4R^9+R^6)}{16(R^3-1)^6}u_2^4 - \frac{15\mu(2R^7+7R^4)}{4(R^3-1)^2})u_2^2 + \frac{2\mu^2}{3}R^5 - \frac{5\mu^2}{3}R^2, \tag{14}$$

$$u_t = u_5 - \frac{15(R^5+2R^2)}{2(R^3-1)^2}u_2u_4 - \frac{45R^2}{4(R^3-1)^2}u_3^2 + \frac{45(R^{10}+22R^7+13R^4)}{4(R^3-1)^4}u_2^2u_3 - \frac{3645(2R^{12}+4R^9+R^6)}{16(R^3-1)^6}u_2^4 - 5\Omega\frac{5R^6+2R^3+2}{9R^4}u_3 + 5\Omega\frac{10R^9+39R^6+36R^3-4}{12R^2(R^3-1)^2}u_2^2 - 5\Omega'\frac{10R^9-3R^6+12R^3+8}{54R^6}u_2 - 5\Omega^2\frac{14R^9+39R^6+24R^3+4}{243R^{10}(R^3-1)^2}. \tag{15}$$

In (14), (15) $R = R(u_1)$ is defined as a solution of the algebraic equation

$$2R^3 - 3u_1R^2 + 1 = 0,$$

and $\Omega(u)$ is any non-constant solution of

$$\Omega'^2 = 4\Omega^3 + c.$$

Substitutions:

$(11)\vert_{\mu_1=\lambda_2,\mu_2=-\lambda_1^2} \to (10)$:	$\log u_1 = \tilde{u}$;
$(12)\vert_{\mu_1=\lambda_1,\mu_2=-\lambda_2^2} \to (9)$:	$-\frac{1}{2}\log u_1 = \tilde{u}$;
$(13) \to (7)$:	$\sqrt{u_1-\mu} = \tilde{u}$;
$(14) \to (9)\vert_{\lambda_2=0,\lambda_1=\mu}$:	$\log R(u_1) = \tilde{u}$;
$(15)\vert_{c=-108\lambda_1^2\lambda_2^2} \to (9)$:	$A(u) + \log R(u_1) = \tilde{u}$,

where $A(u)$ is a non-constant solution of the equation

$$A'^2 = \lambda_2^2 \exp(-4A) - \lambda_1 \exp(2A).$$

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121 Krichever–Novikov equation

$$u_t = u_{xxx} - \frac{3}{2u_x}(u_{xx}^2 - r(u)), \quad r = r_4u^4 + \dots + r_0 \tag{1}$$

➢ Introduced in [1].

➢ This is the only integrable equations of the form $u_t = u_{xxx} + f(u_{xx}, u_x, u)$ which is not related via a differential substitution to KdV equation or linear equation [2]. The Hamiltonian structure and recursion operator were studied in [3, 4].

➢ Bäcklund transformation [5]:

$$u_xv_x = h(u,v), \quad h(u,v) = h(v,u), \quad h_{uuu} = 0, \quad r(u) = h_v^2 - 2hh_{vv}.$$

For example, $r(u) = 4u^3 - g_2u - g_3$ corresponds to

$$h = \frac{1}{A}((uv + \alpha u + \alpha v + g_2/4)^2 - (u + v + \alpha)(4\alpha uv - g_3)), \quad A^2 = r(\alpha)$$

where μ is the parameter of BT.

➢ Nonlinear superposition principle [5] is equivalent to the (Q_4) quad-equation.

➢ Zero curvature representation $U_t = V_x + [V, U]$:

$$U = \frac{1}{u_x}\begin{pmatrix} b & c \\ -a & -b \end{pmatrix}, \quad V = -2U_{xx} + 2[U_x, U] - 3\Big(\frac{u_{xxx}}{u_x} - \frac{3u_{xx}^2}{2u_x^2} + \frac{r(u)}{6u_x^2}\Big)U$$

where a, b, c are defined by $h = a(u)v^2 + 2b(u)v + c(u)$.

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122 Kupershmidt lattice

$$u_{n,t} = u_{n+1,x} + u_0 u_{n,x} + \alpha n u_n u_{0,x}, \quad n = 0, 1, 2, \ldots$$

➢ Dispersionless Lax pair $D_t(L) = A_p L_x - A_x L_p$:

$$A = \frac{p^{\alpha+1}}{\alpha+1} + u_0 p, \quad L = p^\alpha + u_0 + u_1 p^{-\alpha} + u_2 p^{-2\alpha} + \ldots$$

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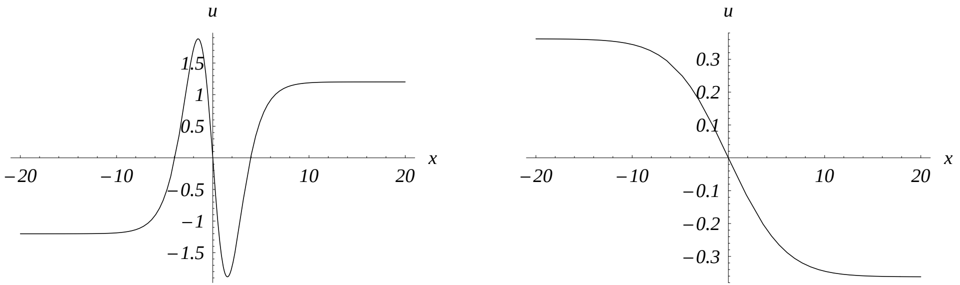
123 Kuramoto–Sivashinsky equation

$$u_t = u_{xxxx} + \mu u_{xx} + u u_x \qquad (1)$$

➢ Introduced in [4]. This equation describes, in particular, the flame propagation.

➢ This equation is nonintegrable. In general, its solutions demonstrate chaotic behaviour. An exact kink-like solution [4] is:

$$u = 120v^3 + 60\left(\frac{\mu}{19} - 2k^2\right)v - c, \quad v = k \tanh k(x - x_0 - ct), \quad 76k^2 = 11\mu \text{ or } 76k^2 = -\mu.$$



$\mu = 1,\ k^2 = 11/76;$ $\quad \mu = -1,\ k^2 = 1/76;$ $\quad (c = 0,\ t = 0,\ x_0 = 0).$

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124 Lagrange top

$$\dot{m} = ap^{\mathsf{T}} - pa^{\mathsf{T}}, \quad \dot{a} = ma, \qquad m \in \mathrm{so}(d), \quad a, p \in \mathbb{R}^d, \quad p = \mathrm{const}$$

This is the rest frame description of the motion in the gravity field of a d-dimensional axially symmetric rigid body with the fixed point on the axis of symmetry.

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125 Lagrange top discrete

$$m_{n+1} = m_n + a_n p^{\mathsf{T}} - p a_n^{\mathsf{T}}, \quad a_{n+1} = (2I + m_n)(2I - m_n)^{-1} a_n, \qquad m_n \in \mathrm{so}(d), \quad a_n, p \in \mathbb{R}^d, \quad p = \mathrm{const}$$

References

[1] Yu.B. Suris. The problem of integrable discretization: Hamiltonian approach. Basel: Birkhäuser, 2003.

126 Lax pair

A nonlinear PDE possesses the ***Lax pair*** if it is equivalent to an equation of the form

$$D_t(L) = [A, L], \quad L = u_n D_x^n + \cdots + u_0, \quad A = a_m D_x^m + \cdots + a_m$$

for some differential operators A, L. The first and simplest example corresponding to Korteweg–de Vries equation is [1]

$$L = -D_x^2 + u, \quad A = 4D_x^3 - 3uD_x - 3D_x u, \quad u_t + u_{xxx} - 6uu_x = 0.$$

References

[1] P.D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.* ***21*** (1968) 467–490.

127 Lax pair dispersionless

A nonlinear PDE possesses the ***dispersionless Lax pair*** if it is equivalent to an equation of the form

$$D_t(L) = \{A, L\} = A_p L_x - A_x L_p, \quad L = L(x,t;p), \quad A = A(x,t;p)$$

for the coefficients of the functions L and A power expansions over parameter p.

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128 Landau–Lifshitz equation

$$S_t=[S,S_{xx}+JS],\quad S\in\mathbb{R}^3,\quad \langle S,S\rangle=1,\quad J=\mathrm{diag}(J_1,J_2,J_3),\quad J_1+J_2+J_3=0 \tag{1}$$

➢ The higher symmetry:

$$S_{t_3}=\big(S_{xx}+\frac{3}{2}\langle S_x,S_x\rangle S\big)_x-\frac{3}{2}\langle S,JS\rangle S_x.$$

➢ The mater-symmetry [2] is local:

$$S_\tau=[s,x(S_{xx}+JS)+S_x]=xS_t+[S,S_x].$$

➢ The ***stereographic projection***

$$S=\frac{1}{1+z\bar z}(z+\bar z,i(z-\bar z),1-z\bar z) \tag{2}$$

brings (1) to the form

$$iz_t=z_{xx}-\frac{2\bar z(z_x^2+r)}{1+z\bar z}+\frac{1}{2}r',\quad 4r=(J_2-J_1)(z^4+1)+6(J_1+J_2)z^2.$$

➢ In the totally anisotropic case $J_i\neq J_k$ the zeroes of the polynomial R are distinct.

➢ In the partially isotropic case one can set, without loss of generality, $J_1=J_2=\pm\frac{1}{3}\delta^2$, $r=\pm\delta^2z^2$. The sign $+/-$ correspond to easy axis/easy plane ferromagnet.

➢ The isotropic case $J=0$, $r=0$ correspond to Heisenberg equation.

➢ The equation remains integrable also for arbitrary 4-th order polynomial r. The complexification $z\to u$, $\bar z\to 1/v$, $t\to it$ yields the system, also known as the Landau–Lifshitz equation:

$$u_t=u_2-\frac{2(u_1^2+r(u))}{u-v}+\frac{1}{2}r'(u),\quad -v_t=v_2-\frac{2(v_1^2+r(v))}{v-u}+\frac{1}{2}r'(v),\quad r^{\mathrm{V}}=0. \tag{3}$$

➢ The higher symmetry:

$$u_{t_3} = u_3 - u_1\Big(\frac{6u_2 + 3r'(u)}{u-v} - \frac{6(u_1^2 + r(u))}{(u-v)^2} - \frac{1}{2}r''(u)\Big),$$

$$v_{t_3} = v_3 - v_1\Big(\frac{6v_2 + 3r'(v)}{v-u} - \frac{6(v_1^2 + r(v))}{(u-v)^2} - \frac{1}{2}r''(v)\Big).$$

➢ The 2×2 zero curvature representation contains the spectral parameter on an elliptic curve. The 4×4 representation polynomial in λ was found in [3].

➢ Bäcklund transformation is defined by Shabat–Yamilov lattice.

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129 Landau–Lifshitz equation, $r = \pm u^2$ (easy axis/easy plane)

$$u_t = u_{xx} - 2\frac{u_x^2 - \delta^2 u^2}{u - v} - \delta^2 u, \quad -v_t = v_{xx} + 2\frac{v_x^2 - \delta^2 v^2}{u - v} - \delta^2 v$$

The case $\delta = 0$ corresponds to the Heisenberg equation.

➢ Bäcklund transformation:

$$u_{n,x} = \frac{2h_n}{u_{n+1} - v_{n+1}} + h_{n,v_{n+1}}, \quad v_{n,x} = \frac{2h_{n-1}}{u_{n-1} - v_{n-1}} - h_{n-1,u_{n-1}},$$

$$h_n = k_n(u_n^2 + v_{n+1}^2) - c_n u_n v_{n+1}, \quad c_n = \frac{2\beta_n^2 - 2\delta\beta_n + \delta^2}{2\beta_n - \delta}, \quad k_n = \frac{\beta_n(\beta_n - \delta)}{2\beta_n - \delta}$$

➢ Zero curvature representation

$$U = \frac{1}{u - v}\begin{pmatrix} \frac{\lambda}{2}(u + v) & -uv \\ \lambda^2 - \delta^2 & -\frac{\lambda}{2}(u + v) \end{pmatrix},$$

$$V = -\lambda U + \frac{1}{(u - v)^2}\begin{pmatrix} -\lambda(uv)_x + \frac{\delta^2}{2}(u^2 - v^2) & u_x v^2 + u^2 v_x \\ (\delta^2 - \lambda^2)(u + v)_x & \lambda(uv)_x - \frac{\delta^2}{2}(u^2 - v^2) \end{pmatrix},$$

$$W_n = h_n^{-1/2}\begin{pmatrix} (\lambda + c_n)v_{n+1} - 2k_n u_n & -u_n v_{n+1} \\ \lambda^2 - \delta^2 & -(\lambda + c_n)u_n + 2k_n v_{n+1} \end{pmatrix}.$$

130 Landau–Lifshitz equation, $r = 1$

$$u_t = u_{xx} - 2\frac{u_x^2 + \delta}{u - v}, \quad -v_t = v_{xx} + 2\frac{v_x^2 + \delta}{u - v}$$

The case $\delta = 0$ corresponds to Heisenberg equation.

➢ Bäcklund transformation:

$$u_{n,x} = \frac{2h_n}{u_{n+1} - v_{n+1}} + h_{n,v_{n+1}}, \quad v_{n,x} = \frac{2h_{n-1}}{u_{n-1} - v_{n-1}} - h_{n-1,u_{n-1}}, \quad h_n = \beta_n(u_n - v_{n+1})^2 + \frac{\delta}{4\beta_n}$$

➢ Zero curvature representation

$$U = \frac{\lambda}{u - v}\begin{pmatrix} \frac{1}{2}(u + v) & -uv - \frac{\delta}{\lambda^2} \\ 1 & -\frac{1}{2}(u + v) \end{pmatrix},$$

$$V = -\lambda U + \frac{\lambda}{(u - v)^2}\begin{pmatrix} -(uv)_x & u_x v^2 + u^2 v_x - \frac{2\delta}{\lambda}(u - v) + \frac{\delta}{\lambda^2}(u + v)_x \\ -(u + v)_x & (uv)_x \end{pmatrix},$$

$$W_n = h_n^{-1/2}\begin{pmatrix} -\lambda v_{n+1} + 2\beta_n(u_n - v_{n+1}) & \lambda u_n v_{n+1} + \frac{\delta}{\lambda} + \frac{\delta}{2\beta_n} \\ -\lambda & \lambda u_n + 2\beta_n(u_n - v_{n+1}) \end{pmatrix}.$$

131 Landau–Lifshitz equation, $r = 0$, Heisenberg equation

$$S_t = [S, S_{xx}], \quad S \in \mathbb{R}^3, \quad \langle S, S\rangle = 1$$

➢ Master-symmetry:

$$\begin{aligned} S_{\tau_0} &= xS_x, \\ S_{\tau_1} &= x[S, S_{xx}] + [S, S_x], \\ S_{\tau_2} &= x(S_{xxx} + 2\langle S_x, S_x\rangle S_x + 2\langle S_x, S_{xx}\rangle S) + 2S_{xx} + 3\langle S_x, S_x\rangle S + S_x D_x^{-1}(\langle S_x, S_x\rangle) \end{aligned}$$

➢ sl_2 version:

$$2S_t = [S, S_{xx}], \quad S \in sl_2, \quad S^2 = 1.$$

➢ Zero curvature representation:

$$U = \lambda S, \quad V = 2\lambda^2 S + \lambda S S_x.$$

➢ Polynomial parametrization:

$$S = \begin{pmatrix} 1 - pq & u \\ q(2 - pq) & pq - 1 \end{pmatrix} \quad \Rightarrow \begin{cases} p_t = p_2 + (p^2 q_1)_x \\ q_t = -q_2 + pq_1^2 \end{cases}$$

➢ Rational parametrization given by the stereographic projection (128.2) brings to

$$u_t = u_{xx} - \frac{2u_x^2}{u - v}, \quad -v_t = v_{xx} + \frac{2v_x^2}{u - v}$$

➢ Bäcklund transformation:

$$u_{n,x} = \frac{\beta_n(v_{n+1} - u_n)(u_n - u_{n+1})}{v_{n+1} - u_{n+1}}, \quad v_{n,x} = \frac{\beta_{n-1}(v_{n-1} - v_n)(v_n - u_{n-1})}{v_{n-1} - u_{n-1}}$$

➢ Zero curvature representation

$$U = \frac{\lambda}{u - v}\begin{pmatrix} \frac{1}{2}(u + v) & -uv \\ 1 & -\frac{1}{2}(u + v) \end{pmatrix}, \quad V = -\lambda U + \frac{\lambda}{(u - v)^2}\begin{pmatrix} -(uv)_x & u_x v^2 + u^2 v_x \\ -u_x - v_x & (uv)_x \end{pmatrix},$$

$$W_n = \frac{1}{u_n - v_{n+1}} \begin{pmatrix} -\lambda v_{n+1} + \beta_n(u_n - v_{n+1}) & \lambda u_n v_{n+1} \\ -\lambda & \lambda u_n + \beta_n(u_n - v_{n+1}) \end{pmatrix}$$

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132 Laplace cascade method

Author: V.V. Sokolov, 25.12.2008

Laplace transformations were introduced in 1873[1, t.9, p.5–68], the following development of the theory was made by Darboux [2, t.2]. The contemporary works are related mainly with the applications in the theory of integrable systems, see e.g. [3] and the section on Liouville type equations. Some generalizations of Laplace method can be found in the papers [4, 5].

1. Laplace invariants and Laplace transformations

The Laplace invariants for linear hyperbolic operator of the form

$$L = \frac{\partial^2}{\partial x \partial y} + a(x,y)\,\frac{\partial}{\partial x} + b(x,y)\,\frac{\partial}{\partial y} + c(x,y) \tag{1}$$

are defined as follows. The operator L can be represented in the following two partially factorized forms

$$L = \Big(\frac{\partial}{\partial x} + b\Big)\Big(\frac{\partial}{\partial y} + a\Big) - h, \qquad h = a_x + ba - c,$$

or

$$L = \Big(\frac{\partial}{\partial y} + a\Big)\Big(\frac{\partial}{\partial x} + b\Big) - k, \qquad k = b_y + ab - c.$$

The functions h and k are invariant with respect to conjugations $L \to s(x,y)Ls(x,y)^{-1}$. The functions h and k are called ***the main left and right Laplace invariants*** of the operator (1).

Equation $L(V) = 0$ is equivalent to

$$\Big(\frac{\partial}{\partial y} + a\Big)V = V_1, \qquad \Big(\frac{\partial}{\partial x} + b\Big)V_1 = hV. \tag{2}$$

If $h \neq 0$, then V_1 satisfies a new hyperbolic equation

$$\Big(\frac{\partial^2}{\partial x \partial y} + a_1(x,y)\frac{\partial}{\partial x} + b_1(x,y)\frac{\partial}{\partial y} + c_1(x,y)\Big)V_1 = L_1(V_1) = 0,$$

where

$$a_1 = a - (\log h)_y, \qquad b_1 = b, \qquad c_1 = a_1 b_1 + b_y - h.$$

In it easy to see that

$$L_1 = \Big(\frac{\partial}{\partial y} + a_1\Big)\Big(\frac{\partial}{\partial x} + b\Big) - h = \Big(\frac{\partial}{\partial x} + b\Big)\Big(\frac{\partial}{\partial y} + a_1\Big) - h_1,$$

where

$$h_1 = a_{1,x} - b_y + h.$$

The main right Laplace invariant k_1 of L_1 coincides with h. Notice that

$$\Big(\frac{\partial}{\partial y} + a_1\Big)L = L_1\Big(\frac{\partial}{\partial y} + a\Big).$$

We say that operator L_1 is obtained as the result of ***Laplace y-transformation*** of the operator L. It follows from (2) that any solution of the equation $L(V) = 0$ produces a solution of the equation $L_1(V_1) = 0$ and vice versa.

If $h_1 \neq 0$, we can apply the same procedure to the operator L_1 and so on. As the result we obtain a chain of operators

$$L_i = \Big(\frac{\partial}{\partial x} + b\Big)\Big(\frac{\partial}{\partial y} + a_i\Big) - h_i = \Big(\frac{\partial}{\partial y} + a_i\Big)\Big(\frac{\partial}{\partial x} + b\Big) - h_{i-1}, \qquad i \in \mathbb{N}, \tag{3}$$

where

$$\Big(\frac{\partial}{\partial y} + a_i\Big)L_{i-1} = L_i\Big(\frac{\partial}{\partial y} + a_{i-1}\Big).$$

The coefficients a_i and the Laplace invariants of these operators are related by the formulas

$$a_i = a_{i-1} - (\log h_{i-1})_y, \qquad h_i = a_{i,x} - b_y + h_{i-1}. \tag{4}$$

Here $a_0 = a$, $h_0 = h$. It follows from these formulas that

$$h_i = 2h_{i-1} - h_{i-2} - (\log h_{i-1})_{xy}, \tag{5}$$

and $k_i = h_{i-1}$. The chain (5) is nothing but the famous integrable Toda lattice.

If $k \neq 0$, then starting from the operator L, we can define another chain of operators $\bar{L}_1, \bar{L}_2, \ldots$ by the relations

$$\Big(\frac{\partial}{\partial x} + b_i\Big)\bar{L}_{i-1} = \bar{L}_i\Big(\frac{\partial}{\partial x} + b_{i-1}\Big).$$

For this chain we have

$$b_i = b_{i-1} - (\log k_{i-1})_x, \qquad a_i = a, \qquad k_i = b_{i,y} - a_x + k_{i-1},$$

where k_i is the main Laplace invariant playing the same role for $\bar{L}_i$ as k for L; another invariant coincides with k_{i-1}. If we denote $h_{-1} = k$ and $h_{-i-1} = k_i$, $i = 1, 2, \ldots$, then we get the complete set of the Laplace invariants h_i, $i \in \mathbb{Z}$. It can be easily verified that all Laplace invariants satisfy (5).

The operator

$$L^{\mathsf{T}} = \frac{\partial^2}{\partial x \partial y} - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y} + (c - a_x - b_y)$$

is called ***adjoint*** to the operator L. It is not difficult to show that the Laplace invariants H_i of the operator L^{T} is related to the Laplace invariants of L by

$$H_n = h_{-n-1}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

2. Laplace integrability

According to (2), we have

$$V_i = \frac{1}{h_i}\Big(\frac{\partial}{\partial x} + b\Big)V_{i+1}$$

and therefore

$$V = \frac{1}{h}\Big(\frac{\partial}{\partial x} + b\Big)\frac{1}{h_1}\Big(\frac{\partial}{\partial x} + b\Big)\cdots\frac{1}{h_{p-1}}\Big(\frac{\partial}{\partial x} + b\Big)V_p,$$

where $L_i(V_i) = 0$. Since

$$\frac{\partial}{\partial x} + b = e^{-\int bdx}\frac{\partial}{\partial x}e^{\int bdx},$$

hence the latter formula can be rewritten as

$$Ve^{\int bdx} = \frac{1}{h}\frac{\partial}{\partial x}\frac{1}{h_1}\frac{\partial}{\partial x}\cdots\frac{1}{h_{p-1}}\frac{\partial}{\partial x}\left(V_p e^{\int bdx}\right). \tag{6}$$

Analogously,

$$Ve^{\int ady} = \frac{1}{k}\frac{\partial}{\partial y}\frac{1}{k_1}\frac{\partial}{\partial y}\cdots\frac{1}{k_{q-1}}\frac{\partial}{\partial y}\left(V_{-q} e^{\int ady}\right),$$

where $\bar{L}_i(V_{-i}) = 0$.

If for some p we have $h_p \equiv 0$, then the chain of operators L_i is terminated. In this case the equation $L_p(V_p) = 0$ can be easily solved. It follows from (3) that

$$\Big(\frac{\partial}{\partial y} + a_p\Big)V_p = Y(y)e^{-\int bdx}$$

or

$$V_p = e^{-\int a_p dy}\Big(X(x) + \int Y(y)e^{\int(a_p dy - bdx)}dy\Big),$$

where X and Y are arbitrary functions of variables x and y correspondingly. Let

$$\alpha = e^{-\int a_p dy}, \qquad \beta = e^{\int a_p dy - bdx},$$

then

$$V_p = \alpha\Big(X + \int Y\beta dy\Big).$$

Substituting this to (6), we get

$$V = A\Big(X + \int Y\beta dy\Big) + A_1\Big(X' + \int Y\frac{\partial\beta}{\partial x}dy\Big) + \cdots + A_p\Big(\frac{d^pX}{dx^p} + \int Y\frac{\partial^p\beta}{\partial x^p}dy\Big),$$

where $A, A_1, \ldots, A_p$ are some given functions of x and y and $X(x)$, $Y(y)$ are arbitrary functions. Thus the general solution of the equation $L(V) = 0$ can be found in quadratures by (6). Notice that this solution is local with respect to the function X but non-local in Y.

Choosing $Y = 0$, we get the following special solution:

$$V = AX + A_1X' + \cdots + A_p\frac{d^pX}{dx^p}. \qquad (7)$$

Thus, if $h_p = 0$, then the equation $L(V) = 0$ has a special solution of the form (7), where $X(x)$ is arbitrary function. The converse statement is also true.

Lemma 1. *Let equation $L(V) = 0$ has a solution of the form (7), where function $X(x)$ is arbitrary. Then an integer m exists such that $0 \le m \le p$ and $h_m = 0$.*

The operator L is called ***Laplace integrable*** if there exist $p \ge 0$ and $q \ge 0$ such that $h_p \equiv 0$ and $k_q \equiv 0$. The concept of the Laplace integrability is crucial for the definition of so called Liouville type nonlinear hyperbolic equations (see also [6, 7] and references therein).

The general solution of the equation $L(V) = 0$ with Laplace integrable operator is local in both arbitrary functions:

$$V = AX + A_1X' + \cdots + A_p\frac{d^pX}{dx^p} + BY + B_1Y' + \cdots + B_q\frac{d^qY}{dy^q}.$$

It is clear that $L^\intercal$ is Laplace integrable if and only if L is Laplace integrable.

3. The matrix case

Consider now operator (1) with coefficients a, b, c being $N \times N$ matrices. A straightforward generalization of all definitions to the matrix case looks as follows. The main Laplace invariants are defined by the formulas

$$h_0 = a_x + ba - c, \qquad h_{-1} = k_0 = b_y + ab - c.$$

Actually, only spectrum of the matrices h_0, h_{-1} is invariant with respect to the transformations $L \to s(x,y)Ls(x,y)^{-1}$. However by an analogy to the scalar case, we prefer to keep the name “invariant” for these matrices.

The matrices h_i for $i > 0$ are recurrently determined from the following system of equations

$$h_{i,y} - h_i a_i + a_{i+1} h_i = 0, \tag{8}$$

$$h_{i+1} = 2h_i + (a_{i+1} - a_i)_x + [b, a_{i+1} - a_i] - h_{i-1}, \tag{9}$$

where $a_0 = a$. Obviously, in the scalar case these formulas coincide with the corresponding ones from section 1. Suppose the matrices h_i and a_i for $i \le k$ are already given. Then we derive a_{k+1} from (8) and after that find h_{k+1} from (9). However if $\det h_k = 0$, then a_{k+1} does not exist at all or it is defined not uniquely but up a matrix α such that $\alpha h_k = 0$. In the latter case, the existence and properties of next Laplace invariants essentially depend on the choice of α. At first glance, such degenerations are very special, but in applications of the Laplace invariants to the Liouville type integrable systems this is a generic case.

To overcome this difficulty, we consider the products $H_i = h_i h_{i-1} \cdots h_1 h_0$. The matrices H_i are called ***generalized Laplace invariants***. For the scalar case the formulas (4) are equivalent to

$$a_i = a - (\log H_{i-1})_y, \qquad h_i = a_{i,x} - b_y + h_{i-1}.$$

These formulas are generalized to the matrix case as follows:

$$H_{i,y} - H_i a + a_{i+1} H_i = 0, \tag{10}$$

$$h_{i+1} = a_{i+1,x} + [b, a_{i+1}] - b_y + h_i, \qquad H_{i+1} = h_{i+1} H_i, \tag{11}$$

and $H_0 = h_0$. To get H_{i+1} we have to solve equation (10) for a_{i+1} and substitute the result into (11). The generalized Laplace invariants K_i are defined by

$$K_{i,x} - K_i b + b_{i+1} K_i = 0, \tag{12}$$

$$k_{i+1} = b_{i+1,y} + [a, b_{i+1}] - a_x + k_i, \qquad K_{i+1} = k_{i+1} K_i, \tag{13}$$

and $K_0 = k_0$.

In a slightly different form the following statement was proved in [6].

Theorem 2. *The generalized Laplace invariant H_m exists and is well-defined if and only if*

$$\Big(\frac{\partial}{\partial y}+a\Big)(\ker H_i) \subset \ker H_i \qquad \textit{and} \qquad \Big(\frac{\partial}{\partial x}+b\Big)(\operatorname{Im} H_i) \subset \operatorname{Im} H_i$$

for all $i < m$. After evident reformulation the statement is valid for the invariants K_i.

As in the scalar case, we have the following theorem.

Theorem 3 (Startsev [8]). *Suppose that the Laplace invariants H_i of operator (1) with matrix coefficients exist and are well defined for all $i \le p$ and $H_p = 0$. Then a differential operator exists*

$$S = \sum_{j=0}^{p} A_j(x, y)\frac{\partial^j}{\partial x^j},$$

where A_j are square matrices and $\det(A_p) \ne 0$, such that $S(f(x))$ is a solution of the system $L(V) = 0$ for any vector-function $f(x)$.

Definition 4. The operator (1) with matrix coefficients is called Laplace integrable if the generalized Laplace invariants H_i and K_i exist, are well defined, and $H_p \equiv 0$, $K_q \equiv 0$ for some $p \ge 0$, $q \ge 0$.

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133 Laurent property

A rational mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ satisfies the ***Laurent property*** if all its iterations $f^k(x)$ are Laurent polynomials on the initial data $x = (x_1, \ldots, x_n)$, that is, the denominator of $f^k(x)$ is at most a monomial in $x_1, \ldots, x_n$.

Example: Somos sequences.

134 Left-symmetric algebra

Author: V.V. Sokolov, 04.07.2006

Left-symmetric algebra $(A, \circ)$ is characterized by identity

$$\mathrm{As}(a, b, c) = \mathrm{As}(b, a, c),$$

where As denotes the associator

$$\mathrm{As}(a, b, c) = (a \circ b) \circ c - a \circ (b \circ c).$$

This class of algebras is obviously a generalization of associative ones for which $\mathrm{As}(X, Y, Z) = 0$. Another example of left-symmetric algebra is given by Euclidean space equipped with the multiplication

$$a \circ b = \langle a, c\rangle b + \langle a, b\rangle c$$

where c is a fixed vector.

The left-symmetric algebras are related with multi-field analogs of Burgers equation.

135 Levi system

$$p_t = p_{xx} + (p^2 + 2pq + 2\beta p)_x, \quad q_t = -q_{xx} + (q^2 + 2pq + 2\beta q)_x$$

➢ Introduced in [1]

➢ Levi system is related to NLS system by Bäcklund transformation

$$uv = pq - q_x, \quad u_x/u = p + q + \beta$$

and to DNLS system by the differential substitution

$$p = b_x/b - ab/2, \quad q = -ab/2.$$

➢ Bäcklund transformation:

$$p_{j,x} = p_j(p_{j+1} - p_j + q_{j+1} - q_j + \beta_{j+1} - \beta_j), \quad q_{j,x} = p_j q_j - p_{j-1} q_{j-1}.$$

➢ Hamiltonian structure for this lattice:

$$\{p_j, q_j\} = -p_j, \quad \{p_j, q_{j+1}\} = p_j, \quad H = \sum \Big(\frac{1}{2} q_j^2 + \beta_j q_j + p_j q_j\Big).$$

➢ Nonlinear superposition principle

$$\begin{aligned} \tilde p_{k-1} &= p_{k-1}\Big(1 - \frac{\beta_k - \beta_{k-1}}{p_{k-1} - q_k}\Big), & \tilde p_k &= p_k\Big(1 + \frac{\beta_k - \beta_{k-1}}{p_{k-1} - q_k - \alpha}\Big), \\ \tilde q_{k-1} &= q_{k-1} + \frac{(\beta_k - \beta_{k-1}) q_k}{p_{k-1} - q_k}, & \tilde q_k &= q_k\Big(1 - \frac{\beta_k - \beta_{k-1}}{p_{k-1} - q_k}\Big), \\ \tilde\beta_{k-1} &= \beta_k, & \tilde\beta_k &= \beta_{k-1} \end{aligned}$$

➢ Zero curvature representation $U_t = V_x + [V, U]$, $W_x = U_1 W - W U$:

$$U = \begin{pmatrix} s - \lambda & -q \\ p & \lambda - s \end{pmatrix}, \quad V = 2(\lambda + s)U + \begin{pmatrix} \frac{1}{2}(p - q) & q \\ p & \frac{1}{2}(q - p) \end{pmatrix}_x, \qquad 2s = p + q + \beta$$

$$W_j = p_j^{-1/2} \begin{pmatrix} p_j & -q_{j+1} \\ p_j & 2\lambda - \beta_{j+1} - q_{j+1} \end{pmatrix}$$

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136 Lie algebra

Lie algebra L is the algebra with the skew-symmetric multiplication $[\cdot,\cdot] : L \times L \to L$ which satisfies the Jacobi identity

$$[a,b] = -[b,a], \quad [a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0.$$

Any associative algebra A gives rise to the Lie algebra A^- with respect to the product $[a,b] = ab - ba$. Any Lie algebra is isomorphic to a subalgebra of some A^-.

137 Lie group

The n-parametric Lie group is a smooth n-dimensional manifold G equipped with the operations of multiplication $G \times G \to G$ and taking the inverse $G \to G$ which are smooth mappings and satisfy the common group axioms (1).

Local Lie groups, Lie groups of transformations

138 Liouville equation

$$u_{xy} = e^u$$

➢ This is a classical example of linearizable equation. The substitution from the linear equation

$$e^u = \frac{2v_x v_y}{v^2} = -2(\log v)_{xy}, \quad v_{xy} = 0$$

yields the formula for the general solution $u = \log\left(\frac{2a'(x)b'(y)}{(a(x)+b(y))^2}\right)$.

➢ Alternatively, the solution can be found from two consistent ODEs

$$u_{xx} - \frac{1}{2}u_x^2 = c(x), \quad u_{yy} - \frac{1}{2}u_y^2 = k(y)$$

with arbitrary functions c, k. An immediate check proves that, indeed,

$$D_y(u_{xx} - \frac{1}{2}u_x^2) = 0, \quad D_x(u_{yy} - \frac{1}{2}u_y^2) = 0$$

in virtue of Liouville equation. These expressions are called y- and ***x-integrals*** respectively.

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139 Liouville type equations

A nonlinear hyperbolic equation of the form

$$u_{xy} = f(x, y, u, u_x, u_y)$$

belongs to Liouville equation type if it possesses some of the following properties:

- its general solution is given by an explicit formula (***Darboux integrability***);
- it is linearizable (C-integrability);
- its symmetry algebra contains arbitrary functions;
- the nontrivial integrals in both characteristics exist;
- the sequence of its Laplace invariants is terminated by zero in both directions (***Laplace integrability***).

It should be stressed that although these properties are in close relation, they are not equivalent and there exist the examples of equations which satisfy only some subset of them. Several classification results are known based on the analysis of these features.

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140 Liouville integrability

Author: A.Ya. Maltsev, 2.10.2009

Consider the Hamiltonian system

$$\dot{x}^i = J^{ij}\frac{\partial H}{\partial x^j} = \{x^i, H\}$$

(summation over repeated indices is assumed) on a manifold $\mathcal{M}^{2n}$ with a non-degenerate Hamiltonian operator $J^{ij}(\mathbf{x})$. This system is called ***integrable in the Liouville sense*** [1] if there exist n first integrals I^ν, $\{I^\nu, H\} = 0$, $\nu = 1, \ldots, n$, such that:

1) the integrals I^ν are functionally independent (rank $||\partial I^\nu/\partial x^i|| = n$);
2) the integrals I^ν commute with each other,

$$\{I^\nu, I^\mu\} = 0, \quad \nu, \mu = 1, \ldots, n;$$

3) the common level surfaces $I^\nu(\mathbf{x}) = \text{const}$, $\nu = 1, \ldots, n$ are compact manifolds in $\mathcal{M}^{2n}$.

The following remarkable facts can be proved under these conditions.

Theorem 1 (Liouville)**.** *Almost all common level surfaces of the first integrals $I^\nu(\mathbf{x}) = \text{const}$, $\nu = 1, \ldots, n$ are n-dimensional tori $\mathbb{T}^n$ embedded in $\mathcal{M}^{2n}$. In the vicinity of every such torus the so called* ***action-angle coordinates*** *$(J^1, \ldots, J^n, \theta^1, \ldots, \theta^n)$ can be introduced such that:*

1) in these coordinates the Poisson bracket has the canonical form (see Darboux theorem)

$$\{J^\alpha, J^\beta\} = 0, \quad \{\theta^\alpha, \theta^\beta\} = 0, \quad \{J^\alpha, \theta^\beta\} = \delta^{\alpha\beta}, \quad \alpha, \beta = 1, \ldots, n;$$

2) on the tori $\mathbb{T}^n$, the coordinates $(\theta^1, \ldots, \theta^n)$ take the values $0 \le \theta^\alpha < 2\pi$, while the coordinates $(J^1, \ldots, J^n)$ are constant;

3) the Hamiltonian function H has constant values on the tori $\mathbb{T}^m$ and depends on the values of $(J^1, \ldots, J^m)$ only: $H = H(J^1, \ldots, J^m)$.

It can be easily deduced then that the tori $\mathbb{T}^n$ give invariant manifolds for the corresponding dynamical system, such that the values of J^α remain constant, while the coordinates θ^α depend linearly on time with some constant frequencies $\omega^\alpha(\mathbf{J})$:

$$\theta^\alpha(t) = \theta_0^\alpha + \omega^\alpha(\mathbf{J})t.$$

The Liouville theorem gives a beautiful description of the global behavior of the trajectories of integrable systems from the geometrical point of view. Thus, the trajectory of an integrable dynamical system gives the irrational covering of some n-dimensional torus $\mathbb{T}^n \subset \mathcal{M}^{2n}$ in generic situation. It is easy to see also that every dynamical system defined by any Hamiltonian function $H = H(\mathbf{J})$ is integrable in the Liouville sense and has the same invariant tori $\mathbb{T}^n$ as the initial one. So, the integrability in Liouville sense implies in fact the existence of infinite number of integrable dynamical systems which commute with each other.

The definition of the integrable system can be generalized also to the case of the Poisson structure of constant rank. Namely, we can say that the Hamiltonian dynamical system is integrable if all Casimir functions are globally defined on $\mathcal{M}^m$ and the restriction of the dynamical system on every common level surface of Casimir functions $N^1 = \text{const}, \ldots, N^{m-2n} = \text{const}$ gives an integrable system in the Liouville sense.

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141 Loop algebra

A ***loop algebra*** is the Lie algebra of the formal Laurent series with the coefficients in some lie algebra L:

$$L(\lambda) = \{u_0\lambda^n + u_1\lambda^{n-1} + u_2\lambda^{n-2} + \dots \mid n \in \mathbb{Z},\ u_i \in L\}.$$

Loop algebras are related with several schemes of construction and solving of integrable systems, from rather simple to more or less universal ones.

As a simplest example consider the Lax equation on $L(\lambda)$ of the form

$$U_{t_n} = [U_n, U], \quad U = u_0 + u_1/\lambda + u_2/\lambda^2 + \dots, \quad U_n = (\lambda^n U)_+ = u_0\lambda^n + u_1\lambda^{n-1} + \dots + u_n. \tag{1}$$

This gives the infinite system of equations for the coefficients

$$\begin{array}{llll}
u_{0,t_0} = 0, & u_{0,t_1} = 0, & u_{0,t_2} = 0, & \dots \\
u_{1,t_0} = [u_0, u_1], & u_{1,t_1} = [u_0, u_2], & u_{1,t_2} = [u_0, u_3], & \dots \\
u_{2,t_0} = [u_0, u_2], & u_{2,t_1} = [u_0, u_3] + [u_1, u_2], & u_{2,t_2} = [u_0, u_4] + [u_1, u_3], & \dots \\
u_{3,t_0} = [u_0, u_3], & u_{3,t_1} = [u_0, u_4] + [u_1, u_3], & u_{3,t_2} = [u_0, u_5] + [u_1, u_4] + [u_2, u_3], & \dots \\
\dots & \dots & \dots & \dots
\end{array}$$

which possesses the properties formulated in the following statement. These properties are not related with the nature of Lie algebra L.

Statement 1. *1) The flows D_{t_n} defined by equations (1) commute, that is $D_{t_n}(D_{t_m}(u_j)) = D_{t_m}(D_{t_n}(u_j))$ for all j, m, n;*

2) the identities $u_{m+1,t_n} = u_{n+1,t_m}$ hold, and this makes possible to introduce the potential $v \in L$ accordingly to the formulas $u_n = v_{t_{n-1}}$;

3) the flow D_τ defined by equation

$$U_\tau = [v, U] - \lambda^2 U_\lambda \quad \Leftrightarrow \quad u_{k,\tau} = [v, u_k] + (k+1)u_{k+1} \tag{2}$$

is the master-symmetry of the hierarchy (1):

$$[D_\tau, D_{t_n}] = nD_{t_{n+1}}.$$

Proof. 3) One has

$$\begin{aligned}[D_\tau, D_{t_n}](U) &= [U_n, U]_\tau - ([v,U] - \lambda^2 U_\lambda)_{t_n} \\ &= [U_{n,\tau} - v_{t_n}, U] + [U_n, [v,U] - \lambda^2 U_\lambda] - [v, [U_n, U]] + \lambda^2 [U_n, U]_\lambda \\ &= [(\lambda^n U_\tau)_+ - u_{n+1} + [U_n, v] + \lambda^2 U_{n,\lambda}, U] \\ &= [-(\lambda^{n+2} U_\lambda)_+ - u_{n+1} + \lambda^2 U_{n,\lambda}, U] = n[U_{n+1}, U],\end{aligned}$$

since

$$\begin{aligned}-(\lambda^{n+2}U_\lambda)_+ - u_{n+1} + \lambda^2 U_{n,\lambda} &= u_1\lambda^n + 2u_2\lambda^{n-1} + \dots + (n+1)u_{n+1} - u_{n+1} \\ &\quad + \lambda^2(nu_0\lambda^{n-1} + (n-1)u_1\lambda^{n-2} + \dots + u_{n-1}) = nU_{n+1}.\end{aligned}$$

■

Various integrable models appear after the concrete choice of the Lie algebra.

Example 2. If L is a finite-dimensional Lie algebra then equations (1) become (1+1)-dimensional integrable systems of NLS or N-wave types. Here the choice of the first coefficient u_0 (which is a constant of motion in virtue of the equations) is of importance.

Example 3. The examples of dispersionless PDE in any dimension appear id L is an infinite-dimensional Lie algebra of the vector fields on some manifold. Let, for instance, L consists of the vector fields on the line, that is u_i are just functions depending on the additional variable x and the commutator is defined by the formula $[u,v] = uv_x - vu_x$ (after identifying $u \leftrightarrow u\partial_x$). Clearly, in the ψ-function language this example corresponds to the auxiliary linear problems of the form $\psi_{t_n} = (\lambda^n + u_1\lambda^{n-1} + \dots + u_n)\psi_x$. The choice $u_0 = 1$ allows to identify D_x and D_{t_0}. In this case equations (1) generate a (2+1)-dimensional dispersionless hierarchy with the simplest representative (in potential form)

$$v_{xt_2} - v_{t_1t_1} + v_x v_{xt_1} - v_{xx} v_{t_1} = 0.$$

More generally, let L be Lie algebra of the vector fields in $\mathbb{R}^N$, that is $u_i = (u_i^1, \dots, u_i^N)$ and the commutator is defined accordingly to the identification $u_i \to \sum u_i^k \partial_{x_k}$. Again, one can assume $D_{t_0} = D_x = D_{x_1}$ without loss of generality, under the choice $u_0 = (1, 0, 0, \dots)$. Then the vector potential $v = (v^1, \dots, v^N)$ satisfies the equation

$$v_{xt_2} - v_{t_1t_1} + [v_x, v_{t_1}] = 0$$

which contain the partial derivatives with respect to $2 + N$ independent variables t_1, t_2 and $x = x_1, \dots, x_N$. The formula (2) leads to the master-symmetries of this equation

$$v_{xT} = [v, v_x] + 2v_{t_1} + t_1(v_{t_1t_1} - [v_x, v_{t_1}]).$$

Interesting reductions correspond to the Lie subalgebras of contact or hamiltonian vector fields. For instance, the choice $N = 2$, $v = (H_p, -H_x)$ leads to 4D-equation

$$H_{xt_2} - H_{t_1t_1} + H_{xt_1}H_{xp} - H_{xx}H_{pt_1} = 0. \tag{3}$$

This construction is not unique. It admits many variations depending on the definition of the element U_n in (1). In particular, these versions are related with different decompositions of Lie algebra L into subalgebras and with the corresponding gradings in $L(\lambda)$. For example, an easy exercise proves, that the choice $U_n = u_0\lambda^n + u_1\lambda^{n-1} + \dots + u_{n-1}\lambda$ lead to the commuting flows as well. In the case $L = sl_2$ these lead to Heisenberg model instead of NLS, and the algebra of Hamiltonian vector fields on the plane leads, instead of (3), to equation

$$H_{t_1t_1} = H_{pt_1}H_{xt_2} - H_{pt_2}H_{xt_1}, \tag{4}$$

which is related to Plebanski equation.

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142 Lorenz system

$$\dot{x} = k(x - y), \quad \dot{y} = rx - y - zx, \quad \dot{z} = xy - bz.$$

This is a famous example of nonintegrable ODE demonstrating strange attractor.

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143 Manakov system

$$u_t = u_{xx} + 2\langle u, v\rangle u, \quad -v_t = v_{xx} + 2\langle u, v\rangle v, \quad u, v \in \mathbb{R}^N$$

This is the first and simplest multifield generalization of NLS equation.

➢ Bäcklund transformation [2, 3]:

$$u_{n,x} = u_{n+1} + \beta_n u_n + \langle u_n, v_{n+1}\rangle u_n, \quad -v_{n,x} = v_{n-1} + \beta_{n-1} v_n + \langle u_{n-1}, v_n\rangle v_n. \tag{1}$$

Third order symmetry:

$$u_{t_3} = u_{xxx} + 3\langle u, v\rangle u_x + 3\langle u_x, v\rangle u, \quad v_{t_3} = v_{xxx} + 3\langle u, v\rangle v_x + 3\langle u, v_x\rangle v.$$

The quantities

$$U = -2\langle u, v\rangle, \quad W = 2\langle u, v_x\rangle - 2\langle u_x, v\rangle$$

satisfy the Kadomtsev–Petviashvili equation [4, 5, 6].

$$4U_{t_3} = U_{xxx} - 6UU_x + 3W_t, \quad W_x = U_t.$$

The quantities

$$F_n = -\langle u_n, v_{n+1}\rangle - \beta_n, \quad P_n = \langle u_n, v_{n+1,x}\rangle - \langle u_{n,x}, v_{n+1}\rangle + \langle u_n, v_{n+1}\rangle^2 - \beta_n^2$$

yield, in virtue of (1), the Miura-type transformation [7]

$$U_{n+1} = U_n + 2F_{n,x}, \quad U_n = F_n^2 - F_{n,x} + P_n, \quad P_{n,x} = F_{n,t}$$

between KP and modified Kadomtsev–Petviashvili equation

$$4F_{t_3} = F_{xxx} - 6(F^2 + P)F_x + 3P_t, \quad P_x = F_t.$$

The variables F, P satisfy the 2D dressing chain

$$F_{n+1,x} + F_{n,x} = F_{n+1}^2 - F_n^2 + P_{n+1} - P_n, \quad P_{n,x} = F_{n,t}.$$

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144 Massive Thirring model

$$iu_x + v + u|v|^2 = 0, \quad iv_t + u + v|u|^2 = 0$$

145 Master-symmetry

An evolutionary equation $u_\tau = K(x, u, u_1, \dots, u_m)$ is called ***master-symmetry*** for equation $u_t = F(x, u, u_1, \dots,$ if the corresponding evolutionary derivatives satisfy the relation $[\nabla_F, [\nabla_F, \nabla_K]] = 0$.

The notion of master-symmetry was introduced by Fokas and Fuchssteiner [1, 2, 3]. The first example appeared actually in [4].

The master symmetries can be introduced through the zero curvature representation with the time-dependent spectral parameters [5, 6, 7].

Master symmetries for many equations (e.g., for the KdV, NLS, and Toda chain equations) are nonlocal. However, the Landau–Lifshitz model, which is an universal equation in the NLS class, has the local master symmetry.

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146 Maxwell–Bloch equation

Reduced Maxwell–Bloch equation:

$$E_t = V, \quad V_x = \omega R + EQ, \quad Q_x = -EV, \quad R_x = -\omega V$$

147 Melnikov system

$$u_t = u_{xxx} + 6uu_x + 3v_{yy} - \langle \phi, \psi \rangle_x, \quad v_x = u, \quad \phi_y = \phi_{xx} + u\phi, \quad -\psi_y = \psi_{xx} + u\psi \tag{1}$$

This multifield generalization [1] of KP equation belongs to the type called equations which self-consistent sources. Equation for the vector ψ coincide with equation of auxiliary linear problem for KP equation and ϕ satisfies the conjugated equation. The choice of the next flow leads to the system

$$\begin{gathered} u_t = u_{xxx} + 6uu_x + 3v_{yy} + 6(\langle \phi, \psi_{xx} \rangle - \langle \phi_{xx}, \psi \rangle + \langle \phi, \psi \rangle_y), \quad v_x = u, \\ 4\phi_t = 4\phi_{xxx} + 6u\phi_x + (3u_x - v - 2\langle \phi, \psi \rangle)\phi, \quad 4\psi_t = 4\psi_{xxx} + 6u\psi_x + (3u_x + v + 2\langle \phi, \psi \rangle)\psi, \end{gathered} \tag{2}$$

which also was introduced in [1]. The similar equation were studied further in [2, 3, 4, 5, 6]. The stationary flow of the system (1) is called Melnikov system as well [7].

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148 Minimal surfaces equation

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0$$

➢ Lagrange function: $L = (1 + u_x^2 + u_y^2)^{1/2}$.

➢ See also Born–Infeld equation

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149 Möbius invariants

Let P_n^m denotes the set of the polynomials on n variables and of degree m on each one. The operations

$$P_4^1 \xrightarrow{\delta_{x_i,x_j}} P_2^2 \xrightarrow{\delta_{x_k}} P_1^4, \qquad \delta_{x,y}(Q) = Q_xQ_y - QQ_{xy}, \quad \delta_x(h) = h_x^2 - 2hh_{xx}$$

are covariant with respect to the Möbius transformations

$$M[f](x_1,\dots,x_n) = (c_1x_1+d_1)^m \dots (c_nx_n+d_n)^m f\left(\frac{a_1x_1+b_1}{c_1x_1+d_1},\dots,\frac{a_nx_n+b_n}{c_nx_n+d_n}\right), \quad f \in P_n^m$$

where $a_id_i - b_ic_i = \Delta_i \neq 0$. More precisely:

$$\delta_{x_i,x_j}(M[Q]) = \Delta_i\Delta_j M[\delta_{x_i,x_j}(Q)], \quad \delta_{x_i}(M[h]) = \Delta_i^2 M[\delta_{x_i}(h)]. \tag{1}$$

The relative invariants of this action for the P_1^4 polynomials $r(x) = r_4x^4 + r_3x^3 + r_2x^2 + r_1x + r_0$ are the coefficients of the Weierstrass normal form $r = 4x^3 - g_2x - g_3$. In terms of the given polynomial, they are [1]

$$\begin{aligned} g_2(r,x) &= \frac{1}{48}(2rr^{IV} - 2r'r''' + (r'')^2) = \frac{1}{12}(12r_0r_4 - 3r_1r_3 + r_2^2), \\ g_3(r,x) &= \frac{1}{3456}(12rr''r^{IV} - 9(r')^2r^{IV} - 6r(r''')^2 + 6r'r''r''' - 2(r'')^3) \\ &= \frac{1}{432}(72r_0r_2r_4 - 27r_1^24 + 9r_1r_2r_3 - 27r_0r_3^2 - 2r_2^3). \end{aligned}$$

Under the Möbius change of $x = x_1$ these quantities are multiplied by simple factors:

$$g_k(M[r],x) = \Delta_1^{2k} g_k(r,x), \quad k = 2,3.$$

For the biquadratic polynomial $h \in P_2^2$,

$$h(x,y) = h_{22}x^2y^2 + h_{21}x^2y + h_{20}x^2 + h_{12}xy^2 + h_{11}xy + h_{10}x + h_{02}y^2 + h_{01}y + h_{00}, \tag{2}$$

the relative invariants are

$$\begin{aligned} i_2(h,x,y) &= 2hh_{xxyy} - 2h_x h_{xyy} - 2h_y h_{xxy} + 2h_{xx}h_{yy} + h_{xy}^2 = \\ &= 8h_{00}h_{22} - 4h_{01}h_{21} - 4h_{10}h_{12} + 8h_{02}h_{20} + h_{11}^2, \\ i_3(h,x,y) &= \frac{1}{4}\det\begin{pmatrix} h & h_x & h_{xx} \\ h_y & h_{xy} & h_{xxy} \\ h_{yy} & h_{xyy} & h_{xxyy}\end{pmatrix} = \det\begin{pmatrix} h_{22} & h_{21} & h_{20} \\ h_{12} & h_{11} & h_{10} \\ h_{02} & h_{01} & h_{00}\end{pmatrix} = -\frac{1}{4}\delta_{x,y}(\delta_{x,y}(h))/h. \end{aligned}$$

Under the Möbius change of $x = x_1$ and $y = x_2$,

$$i_k(M[h],x,y) = \Delta_1^k \Delta_2^k i_k(h,x,y), \quad k = 2,3.$$

The following properties of the operations $\delta_{x,y}$, δ_x are proved straightforwardly.

Lemma 1. *The following identities hold for any affine-linear polynomial* $Q(x,y,u,v) \in P_4^1$ *and any bi-quadratic polynomial* $h(x,y) \in P_2^2$*:*

$$\delta_u(\delta_{xy}(Q)) = \delta_y(\delta_{xu}(Q)), \tag{3}$$

$$i_k(\delta_{xy}(Q),u,v) = i_k(\delta_{uv}(Q),x,y), \quad k = 2,3, \tag{4}$$

$$g_k(\delta_x(h),y) = g_k(\delta_y(h),x), \quad k = 2,3. \tag{5}$$

Denote $Q^{ij} = Q^{ji} = \delta_{x_k,x_l}(Q)$ where $\{i,j,k,l\} = \{1,2,3,4\}$. Lemma 1 implies the commutativity of the diagram

$$\begin{array}{ccccc} r_4(x_4) & \xleftarrow{\delta_{x_3}} & Q^{34}(x_3,x_4) & \xrightarrow{\delta_{x_4}} & r_3(x_3) \\ \delta_{x_1}\big\uparrow & & \big\uparrow\delta_{x_1,x_2} & & \big\uparrow\delta_{x_2} \\ Q^{14}(x_1,x_4) & \xleftarrow{\delta_{x_2,x_3}} & Q(x_1,x_2,x_3,x_4) & \xrightarrow{\delta_{x_1,x_4}} & Q^{23}(x_2,x_3) \\ \delta_{x_4}\big\downarrow & & \big\downarrow\delta_{x_3,x_4} & & \big\downarrow\delta_{x_3} \\ r_1(x_1) & \xleftarrow{\delta_{x_2}} & Q^{12}(x_1,x_2) & \xrightarrow{\delta_{x_1}} & r_2(x_2) \end{array} \tag{6}$$

Moreover, the biquadratic polynomials on the opposite edges have the same invariants i_2, i_3, and invariants g_2, g_3 coincide for all r_i. This diagram can be completed by the polynomials Q^{13}, Q^{24} corresponding to the diagonals (so that the graph of the tetrahedron appears). The polynomials Q^{ij} satisfy a number of important identities.

Lemma 2. *The following identities hold:*

$$4i_3(Q^{12},x_1,x_2)Q^{14} = \det\begin{pmatrix} Q^{12} & Q^{12}_{x_1} & S \\ Q^{12}_{x_2} & Q^{12}_{x_1x_2} & S_{x_2} \\ Q^{12}_{x_2x_2} & Q^{12}_{x_1x_2x_2} & S_{x_2x_2} \end{pmatrix}, \quad S = Q^{23}_{x_3x_3}Q^{34} - Q^{23}_{x_3}Q^{34}_{x_3} + Q^{23}Q^{34}_{x_3x_3}, \tag{7}$$

$$Q^{12}Q^{34} - Q^{14}Q^{23} = PQ, \quad P = \det\begin{pmatrix} Q & Q_{x_1} & Q_{x_3} \\ Q_{x_2} & Q_{x_1x_2} & Q_{x_2x_3} \\ Q_{x_4} & Q_{x_1x_4} & Q_{x_3x_4} \end{pmatrix} \in P^1_4, \tag{8}$$

$$\frac{2Q_{x_1}}{Q} = \frac{Q^{12}_{x_1}Q^{34} - Q^{14}_{x_1}Q^{23} + Q^{23}Q^{34}_{x_3} - Q^{23}_{x_3}Q^{34}}{Q^{12}Q^{34} - Q^{14}Q^{23}}. \tag{9}$$

The identity (7) shows that Q^{14} can be expressed through three other polynomials (provided $i_3(Q^{12}) \neq 0$). The identity (8) defines Q as one of the factor in the simple expression builded from Q^{ij}. Finally, differentiating (9) with respect to x_2 or x_4 brings to the relation of the form $Q^2 = F[Q^{12}, Q^{23}, Q^{34}, Q^{14}]$, where F is a rational expression on Q^{ij} and their derivatives. Therefore, if the polynomials on the edges are known (three is enough) then Q is found explicitly.

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150 Monge–Ampére equation

$$A(u_{xx}u_{yy} - u_{xy}^2) + Bu_{xx} + Cu_{xy} + Du_{yy} + E = 0$$

with the coefficients depending on x, y, u, u_x, u_y.

This class of equations is invariant with respect to contact transformations. In some very special cases this allows to obtain the general solution in parametric form. In particular, the homogeneous Monge–Ampére equation

$$u_{xx}u_{yy} = u_{xy}^2$$

trivializes under the transformation... This provides the general solution in parametric form:

$$u(x, y) = xt + yf(t) + g(t), \quad x + yf'(t) + g'(t) = 0.$$

Analogously, the general solution of the equation

$$u_{xx}u_{yy} - u_{xy}^2 + a^2 = 0$$

is given by

$$u(x, y) = \frac{(s+t)(f'(s) - g'(t)) - 2f(s) + 2g(t)}{4a}, \quad x = \frac{s-t}{2a}, \quad y = \frac{f'(s) - g'(t)}{2a}.$$

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151 Multi-field equations

Author: V.V. Sokolov, 04.07.2006

1. Jordan algebras and generalizations of KdV equation

Consider multi-component generalizations

$$u^i_t = u^i_{xxx} + C^i_{jk} u^j u^k_x \tag{1}$$

of the Korteweg-de Vries equation.

Let us regard C^i_{jk} as the structural constants of an (noncommutative and nonassociative) algebra J and rewrite (1) in the form

$$U_t = U_{xxx} + U \circ U_x,$$

where $U(x,t)$ is a J-valued function.

A system of equations (1) is called *irreducible* if it cannot be reduced to the block-triangular form by an appropriate linear transformation.

Theorem 1 (Svinolupov [10]). *The irreducible KdV-type system (1) possesses higher symmetries if and only if C^i_{jk} are structural constants of a simple Jordan algebra.*

In particular, Jordan algebras given in Example 1 bring to the matrix KdV-equation

$$U_t = U_{xxx} + UU_x + U_xU$$

and the vector KdV equation

$$u_t = u_{xxx} + \langle C, u\rangle u_x + \langle C, u_x\rangle u - \langle u, u_x\rangle C.$$

2. Left-symmetric algebras and generalizations of Burgers equation

Theorem 2. *The multi-component generalization of the Burgers equation*

$$u_t^i = u_{xx}^i + 2C_{jk}^i u^k u_x^j + A_{jkm}^i u^k u^j u^m, \quad i,j,k = 1,\ldots,N$$

is integrable if and only if

$$3A_{jkm}^i = C_{jr}^i C_{km}^r + C_{kr}^i C_{mj}^r + C_{mr}^i C_{jk}^r - C_{rj}^i C_{km}^r - C_{rk}^i C_{mj}^r - C_{rm}^i C_{jk}^r$$

and C_{jk}^i are structural constants of a left-symmetric algebra A.

The coordinate-free form of these integrable equations is

$$u_t = u_{xx} + 2u \circ u_x + u \circ (u \circ u) - (u \circ u) \circ u$$

where $\circ$ denote the multiplication in A.

In particular, the following matrix equation is integrable

$$U_t = U_{xx} + 2UU_x.$$

Another example is the vector Burgers equation

$$u_t = u_{xx} + 2\langle u, u_x\rangle C + 2\langle u, C\rangle u_x + \|u\|^2\langle u, C\rangle C - \|C\|^2\|u\|^2 u$$

where C is a constant vector.

3. Jordan triple systems and generalizations of mKdV and NLS equations

Theorem 3. *If C_{jkm}^i are structural constants of a Jordan triple system then the mKdV-type system*

$$u_t^i = u_{xxx}^i + C_{jkm}^i u^k u^j u_x^m, \quad i,j,k = 1,\ldots,N,$$

the NLS-type system

$$u^i_t = u^i_{xx} + C^i_{jkm} u^j v^k u^m, \quad v^i_t = -v^i_{xx} - C^i_{jkm} v^j u^k v^m, \quad i, j, k = 1, \dots, N$$

and the DNLS-type system

$$u^i_t = u^i_{xx} + C^i_{jkm} (u^j v^k u^m)_x, \quad v^i_t = -v^i_{xx} - C^i_{jkm} (v^j u^k v^m)_x, \quad i, j, k = 1, \dots, N$$

possess higher symmetries.

The algebraic forms of these system are, respectively,

$$u_t = u_{xxx} + \{u, u, u_x\},$$
$$u_t = u_{xx} + 2\{u, v, u\}, \quad v_t = -v_{xx} - 2\{v, u, v\},$$
$$u_t = u_{xx} + 2\{v, u, v\}_x, \quad v_t = -v_{xx} - 2\{u, v, u\}_x.$$

In particular, the simple Jordan triple systems (92.3), (92.4) and (92.5) correspond to the following integrable vector and matrix generalizations of KdV equation

$$u_t = u_{xxx} + \|u\|^2 u_x, \quad u \in \mathbb{R}^N$$
$$u_t = u_{xxx} + \|u\|^2 u_x + \langle u, u_x \rangle u, \quad u \in \mathbb{R}^N,$$
$$U_t = U_{xxx} + U^2 U_x + U_x U^2, \quad U \in \mathrm{Mat}_N .$$

The vector generalizations of NLS are of the form

$$u_t = u_{xx} + 2\langle u, v \rangle u, \quad v_t = -v_{xx} - 2\langle v, u \rangle v$$

and

$$u_t = u_{xx} + 4\langle u, v \rangle u - 2\|u\|^2 v, \quad v_t = -v_{xx} - 4\langle v, u \rangle v + 2\|v\|^2 u.$$

4. Deformations of Jordan triple systems

Consider now non-polynomial integrable equations such as

$$u_{xy} = \frac{u_x u_y}{u},$$

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x},$$

$$u_t = u_{xx} - \frac{2}{u+v}u_x^2, \quad v_t = -v_{xx} + \frac{2}{u+v}v_x^2.$$

How to generalize these equations to the multi-component case? What is u^{-1}?

In the scalar case we can define x^{-1} as a solution of ODE $y' = -y^2$.

Let $\{X, Y, Z\}$ be a Jordan triple system, $\phi(u)$ be a solution of the following overdetermined consistent system

$$\frac{\partial \phi}{\partial u^k} = -\{\phi, e_k, \phi\}, \quad k = 1, \dots, N, \tag{2}$$

where $e_1, \dots, e_N$ is a basis of the Jordan triple system and $u = u^i e_i$.

In the matrix case one of the solutions is

$$\phi(U) = U^{-1}.$$

For the vector Jordan triple system (92.4)

$$\phi(u) = \frac{u}{\|u\|^2}.$$

An analog of u^{-1} is well-known in the theory of the Jordan triple systems. Let us define a linear operator P_X by the formula $P_X(Y) = \{X, Y, X\}$. Then, by definition, $u^{-1} = P_u^{-1}(u)$.

Introduce the notation

$$\alpha_u(X, Y) = \{X, \phi(u), Y\}, \qquad \sigma_u(X, Y, Z) = \{X, \{\phi(u), Y, \phi(u)\}, Z\}.$$

Class 1. For any Jordan triple system the Jordan chiral field equation

$$u_{xy} = \alpha_u(u_x, u_y)$$

is integrable. The terminology originates from the matrix case which corresponds to the equation of the principal chiral field

$$u_{xy} = \frac{1}{2}(u_x u^{-1} u_y + u_y u^{-1} u_x), \quad u \in GL_N.$$

Class 2. The following equation

$$u_t = u_{xxx} - 3\alpha_u(u_x, u_{xx}) + \frac{3}{2}\sigma_u(u_x, u_x, u_x).$$

is integrable. Matrix and vector equations have the following form:

$$u_t = u_{xxx} - \frac{3}{2}u_x u^{-1} u_{xx} - \frac{3}{2}u_{xx} u^{-1} u_x + \frac{3}{2}u_x u^{-1} u_x u^{-1} u_x,$$

$$u_t = u_{xxx} - \frac{3\langle u, u_x\rangle}{\|u\|^2}u_{xx} - 3\frac{\langle u, u_{xx}\rangle}{\|u\|^2}u_x + 3\frac{\langle u_x, u_{xx}\rangle}{\|u\|^2}u - \frac{3}{2}\frac{\|u_x\|^2}{\|u\|^2}u_x + 6\frac{\langle u, u_x\rangle^2}{\|u\|^4}u_x - 3\frac{\langle u, u_x\rangle\|u_x\|^2}{\|u\|^4}u,$$

$$u_t = u_{xxx} - \frac{3}{2}\frac{\langle C, u_x\rangle}{\langle C, u\rangle}u_{xx} - \frac{3}{2}\frac{\langle C, u_{xx}\rangle}{\langle C, u\rangle}u_x + \frac{3}{2}\frac{\langle C, u_x\rangle^2}{\langle C, u\rangle^2}u_x.$$

Class 3. The following integrable equations

$$v_t = v_{xxx} - \frac{3}{2}\alpha_{v_x}(v_{xx}, v_{xx})$$

are related to ones of Class 2 by the potentiation $u = v_x$. The matrix equation is

$$U_t = U_{xxx} - \frac{3}{2}U_{xx}U_x^{-1}U_{xx}.$$

Vector equations are of the form:

$$u_t = u_{xxx} - 3\frac{\langle u_x, u_{xx}\rangle}{\|u_x\|^2}u_{xx} + \frac{3}{2}\frac{\|u_{xx}\|^2}{\|u_x\|^2}u_x$$

and

$$u_t = u_{xxx} - \frac{3}{2}\frac{\langle C, u_{xx}\rangle}{\langle C, u_x\rangle}u_{xx}.$$

Class 4. The scalar representative of this class is the Heisenberg model

$$u_t = u_{xx} - \frac{2}{u+v}u_x^2, \quad v_t = -v_{xx} + \frac{2}{u+v}v_x^2.$$

The following coupled equation

$$u_t = u_{xx} - 2\alpha_{u+v}(u_x, u_x), \quad v_t = -v_{xx} + 2\alpha_{u+v}(v_x, v_x)$$

is integrable. This equation has a higher symmetry of the form

$$u_t = u_{xxx} - 6\alpha_{u+v}(u_x, u_{xx}) + 6\sigma_{u+v}(u_x, u_x, u_x), \quad v_t = v_{xxx} - 6\alpha_{u+v}(v_x, v_{xx}) + 6\sigma_{u+v}(v_x, v_x, v_x).$$

The matrix equation from this class is of the form

$$u_t = u_{xx} - 2u_x(u+v)^{-1}u_x, \quad v_t = -v_{xx} + 2v_x(u+v)^{-1}v_x$$

and one of the two vector equations is

$$u_t = u_{xx} - 4\frac{\langle u_x, u+v\rangle}{\|u+v\|^2}u_x + 2\frac{\|u_x\|^2}{\|u+v\|^2}(u+v), \quad v_t = -v_{xx} + 4\frac{\langle v_x, u+v\rangle}{\|u+v\|^2}v_x - 2\frac{\|v_x\|^2}{\|u+v\|^2}(u+v).$$

5. Integrable equations of geometric type

Consider multi-component systems of the form

$$u_t^i = u_{xxx}^i + a_{jk}^i(\vec{u})u_x^j u_{xx}^k + b_{jks}^i(\vec{u})u_x^j u_x^k u_x^s.$$

This class is invariant under point transformations $\vec{v} = \vec{\Psi}(\vec{u})$. Under these transformations, the functions $a_{jk}^i(\vec{u})$ are transformed as components of an affine connection Γ.

It is convenient to rewrite the system as

$$u_t^i = u_{xxx}^i + 3\alpha_{jk}^i u_x^j u_{xx}^k + \left(\frac{\partial \alpha_{km}^i}{\partial u^j} + 2\alpha_{jr}^i \alpha_{km}^r - \alpha_{rj}^i \alpha_{km}^r + \beta_{jkm}^i\right) u_x^j u_x^k u_x^m,$$

where $\beta_{jkm}^i = \beta_{kjm}^i = \beta_{mkj}^i$, i.e.

$$\beta(X, Y, Z) = \beta(Y, X, Z) = \beta(X, Z, Y)$$

for any vectors X, Y, Z. The set of functions β_{jkm}^i are transformed just as components of a tensor.

Let R and T be the curvature and torsion tensors of Γ.

In order to formulate classification results, we introduce the following tensor:

$$\sigma(X, Y, Z) = \beta(X, Y, Z) - \frac{1}{3}\delta(X, Y, Z) + \frac{1}{3}\delta(Z, X, Y),$$

where

$$\delta(X, Y, Z) = T(X, T(Y, Z)) + R(X, Y, Z) - \nabla_X(T(Y, Z)).$$

It follows from the Bianchi identity that

$$\sigma(X, Y, Z) = \sigma(Z, Y, X).$$

Theorem 4. *The system is integrable if and only if*

$$\nabla_X[R(Y, Z, V)] = R(Y, X, T(Z, V)),$$

$$\nabla_X \left[\nabla_Y (T(Z,V)) - T(Y, T(Z,V)) - R(Y,Z,V)\right] = 0,$$
$$\nabla_X (\sigma(Y,Z,V)) = 0,$$
$$T(X, \sigma(Y,Z,V)) + T(Z, \sigma(Y,X,V)) + T(Y, \sigma(X,V,Z)) + T(V, \sigma(X,Y,Z)) = 0,$$

and

$$\sigma(X, \sigma(Y,Z,V), W) - \sigma(W, V, \sigma(X,Y,Z)) + \sigma(Z, Y, \sigma(X,V,W)) - \sigma(X, V, \sigma(Z,Y,W)) = 0.$$

If $T = 0$, we have the symmetric space with covariantly constant deformation of a triple Jordan system.

In the case $T \neq 0$, a generalization of the symmetric spaces gives rise. We do not know whether such affine connected spaces have been considered by geometers.

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152 Multi-Hamiltonian structure

Author: A.Ya. Maltsev, 2.10.2009

The Poisson brackets $\{\cdot,\cdot\}_1$, $\{\cdot,\cdot\}_2$ form the consistent pair [1] if the linear pencil $\{\cdot,\cdot\}_1 + \alpha\{\cdot,\cdot\}_2$ defines a Poisson bracket as well. Obviously, only the Jacobi identity needs the check, moreover, it is easy to show that if it holds at some particular value $\alpha \neq 0$ then it holds for any α.

The equation (ODE, DΔE or PDE) possesses the bi-Hamiltonian structure if it can be represented in the form

$$u_t = \{u, H_1\}_1 = \{u, H_2\}_2$$

with the consistent pair of brackets. Analogously, the tri-Hamiltonian structure is defined by equations

$$u_t = \{u, H_1\}_1 = \{u, H_2\}_2 = \{u, H_3\}_3$$

where the brackets form the consistent triple, that is the operation $\{\cdot,\cdot\}_1 + \alpha\{\cdot,\cdot\}_2 + \beta\{\cdot,\cdot\}_3$ is a Poisson bracket for all α, β.

Consider in more details the finite-dimensional situation corresponding to the pair of Poisson brackets on $\mathcal{M}^n$ defined by the structure matrices $J_1^{ij}(\mathbf{x})$ and $J_2^{ij}(\mathbf{x})$. By the definition, they are compatible if the tensor

$$J_1^{ij} + \lambda J_2^{ij}$$

defines a Poisson bracket on $\mathcal{M}^n$ for every value of λ. It is not difficult to check that this amounts to the Jacobi identities for both J_1^{ij} and J_2^{ij} plus the condition that the ***Schouten bracket***

$$\{J_1, J_2\}^{ijk}_{Sch} = J_1^{iq}\frac{\partial J_2^{jk}}{\partial x^q} + J_1^{jq}\frac{\partial J_2^{ki}}{\partial x^q} + J_1^{kq}\frac{\partial J_2^{ij}}{\partial x^q} + J_2^{iq}\frac{\partial J_1^{jk}}{\partial x^q} + J_2^{jq}\frac{\partial J_1^{ki}}{\partial x^q} + J_2^{kq}\frac{\partial J_1^{ij}}{\partial x^q}$$

vanishes identically on $\mathcal{M}^n$.

If both J_1^{ij} and J_2^{ij} are non-degenerate then the recursion operator

$$R^i_j = J_2^{iq}||J_1^{-1}||_{qj}$$

can be defined and has non-zero eigenvalues.

All the eigenvalues of R^i_j are double-degenerated and generically R^i_j has $m = n/2$ distinct eigenvalues $(\lambda_1, \ldots, \lambda_m)$. The eigenvalues of R^i_j as the functions of $\mathbf{x}$ give the set of Hamiltonian functions which commute with each other in both the Poisson structures J_1 and J_2. In the case when the m eigenvalues of R^i_j are functionally independent and have compact common level surfaces every eigenvalue of R^i_j generates an integrable system in both of the two compatible Poisson structures. We get then the infinite set of integrable systems generated by Hamiltonian functions depending on eigenvalues of R^i_j only according to the first or the second Poisson structure J_1 or J_2. Both these sets of integrable systems coincide with each other and all the systems commute with each other as the dynamical systems.

The recursion operator R^i_j gives a possibility to define the higher Poisson brackets J^{ij}_N, $N \geq 1$ according to the formula

$$J^{ij}_N = \left(R^{N-1}\right)^i_q J^{qj}_1.$$

All brackets J^{ij}_N are compatible with each other and give the Poisson structures for the integrable systems considered above. (In fact the same is true also for $N \leq 0$). This construction permits to introduce the Hamiltonian hierarchy of integrable systems (see integrable hierarchy) for every Hamiltonian function $H_1(\lambda_1, \ldots, \lambda_m)$ according to the formula

$$\frac{dx^i}{dt_N} = J^{ij}_N \frac{\partial H_1}{\partial x^j}.$$

All the systems from the hierarchy commute with each other and are Hamiltonian with respect to every bracket from the constructed set. It is possible to introduce also the hierarchy of Hamiltonian functions H_N such that

$$\frac{dx^i}{dt_N} = J^{ij}_1 \frac{\partial H_N}{\partial x^j}.$$

All functions H_N depend on eigenvalues $(\lambda_1, \ldots, \lambda_m)$ only and give the conservation laws for the dynamical systems (only m of them are functionally independent).

The construction described above plays the basic role in the case of compatible brackets of constant rank. In this case the integrable systems arise in many examples as the hierarchies generated by the Casimir functions of the first bracket considered as the Hamiltonian functions in the second Poisson structure. All the Hamiltonian functions H_N^ν of corresponding hierarchies are connected by the relations

$$J_1^{ij}\frac{\partial H_{N+1}^\nu}{\partial x^j}=J_2^{ij}\frac{\partial H_N^\nu}{\partial x^j}\,,\qquad N\geq 1$$

where $H_1^\nu=N^\nu$ are the annihilators (Casimir functions) of the bracket J_1^{ij}, $\nu=1,\dots,n-2s$.

It can be proved in this case that all the functionals H_N^ν commute with each other in both Poisson structures J_1, J_2 and all arising dynamical systems mutually commute. For the integrability in the Liouville sense we have to require then that the set $\{H_N^\nu\}$ gives s functionally independent functions after the restriction on every common level surface $N^\nu=\text{const}$, $\nu=1,\dots,n-2s$, which have compact common level surfaces on these manifolds. The global existence of the Casimir functions N^1, …, N^{n-2s} on the manifold $\mathcal{M}^n$ is also assumed in this situation.

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153 Neumann system

$$\ddot{u} = -Ju + (\langle u, Ju\rangle - \langle \dot{u}, \dot{u}\rangle)u, \quad u \in \mathbb{R}^d, \quad |u| = 1, \quad J = \mathrm{diag}(J_1, \dots, J_d)$$

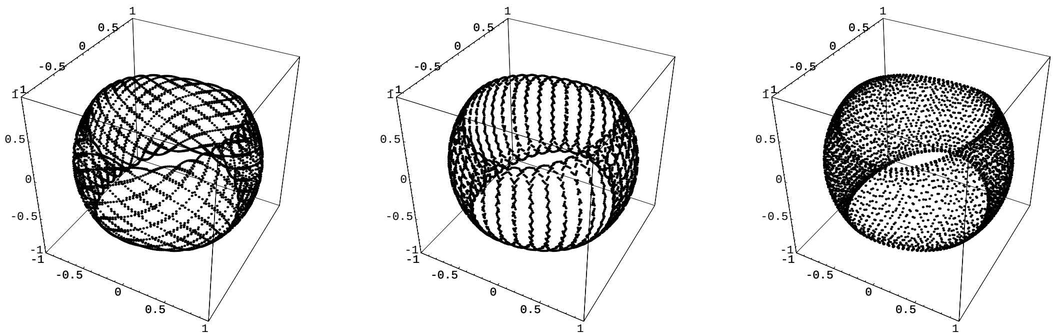
➢ This system, introduced in [1], describes the motion of a particle on the sphere in the quadratic potential $\frac{1}{2}\langle u, Ju\rangle$. It describes also a certain class of exact solutions of Landau–Lifshitz equation [2].

➢ Several discrete-time integrable systems on the sphere are known, of the general form

$$u_{n+1} = F(u_n, u_{n-1}; K), \quad u_n \in \mathbb{R}^d, \quad |u_n| = 1, \quad K = \mathrm{diag}(K_1, \dots, K_d),$$

which have the Neumann system as the continuous limit [3]. Although these discretizations possess the different sets of the invariants which are not equivalent even at $d = 3$, the corresponding dynamics is very similar. The plots below show the evolution of the same initial data (at the same choice of the parameter matrix K).

Veselov discretization Ragnisco discretization Adler discretization



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154 Neumann system, Adler discretization

$$\frac{u_{n+1}+u_n}{1+\langle u_n,u_{n+1}\rangle}+\frac{u_n+u_{n-1}}{1+\langle u_n,u_{n-1}\rangle}=\frac{2Ku_n}{\langle u_n,Ku_n\rangle},\qquad u_n\in\mathbb{R}^d,\quad |u_n|=1,\quad K=\operatorname{diag}(K_1,\dots,K_d)$$

➢ Continuous limit: $u_n=u(2\varepsilon n)$, $K=I-\varepsilon^2 J$.

➢ The invariants:

$$I_1=\frac{\langle Ku_n,u_{n+1}\rangle}{1+\langle u_n,u_{n+1}\rangle},\quad I_2=\frac{\langle K^{-1}(u_n+u_{n+1}),(u_n+u_{n+1})\rangle}{(1+\langle u_n,u_{n+1}\rangle)^2},\ \dots$$

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155 Neumann system, Ragnisco discretization

$$\frac{u_{n+1}}{\langle u_n, u_{n+1}\rangle} - 2u_n + \frac{u_{n-1}}{\langle u_n, u_{n-1}\rangle} = -K^{-2}u_n + \langle u_n, K^{-2}u_n\rangle u_n, \qquad u_n \in \mathbb{R}^d, \quad |u_n| = 1, \quad K = \operatorname{diag}(K_1, \dots, K_d)$$

Continuous limit: $u_n = u(\varepsilon n)$, $K^{-2} = \varepsilon^2 J$.

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156 Neumann system, Veselov discretization

$$u_{n+1}+u_{n-1}=\frac{2\langle Ku_n,u_{n-1}\rangle}{\langle K^2u_n,u_n\rangle}Ku_n,\qquad u_n\in\mathbb{R}^d,\quad |u_n|=1,\quad K=\mathrm{diag}(K_1,\dots,K_d)$$

Alias: stationary Heisenberg spin chain

➢ Continuous limit: $u_n=u(\varepsilon n)$, $K^{-2}=I+\varepsilon^2J$.

➢ The invariants:

$$I_1=\langle Ku_n,u_{n+1}\rangle,\quad I_2=\langle K^{-2}u_n,u_n\rangle+\langle K^{-2}u_{n+1},u_{n+1}\rangle-\langle K^{-1}u_n,u_{n+1}\rangle^2,\ \dots$$

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157 Noether theorem

Author: A.B. Shabat, 27.02.2007

By a conservation law for a differential system

$$\omega^a(x, u, u_i, \dots) = 0, \quad i = 1, \dots, m+1, \quad a = 1, \dots, n,$$
$$u_i^a = \partial u^a / \partial x^i, \quad x^i = (x^1, x^2, \dots, x^m, t)$$

is meant a continuity equation

$$\sum D_i K_i \equiv 0, \quad K_i = K_i(x, u, u_j, \dots), \quad i, j = 1, \dots, m+1,$$

which is satisfied for any solutions of the original system. Each conservation law is defined up to an equivalence transformation $K_i \to K_i + P_i$, $\sum D_i P_i \equiv 0$. Two conservation laws belong to the same equivalence class if they differ by a trivial conservation law. For trivial conservation laws the components of the vector K_i vanish on the solutions: $K_i = 0$, $(i, j = 1, \dots, m+1)$, or the continuity equation is satisfied in the whole space: $\sum D_i K_i \equiv 0$; first and second types of triviality, respectively.

We consider functions $u = u(x)$ defined on a region D of $(m+1)$-dimensional space-time. Let

$$S = \int_D L(x^i, u^a, u_i^a, \dots) d^{m+1} x$$

be the action functional, where L is the Lagrangian density. Then the equations of motion are

$$E^a(L) = \omega^a(x, u, u_i, u_{ij} \dots) = 0, \quad i, j = 1, \dots, m+1, \quad a = 1, \dots, n$$

where E is the Euler–Lagrange operator

$$E^a = \frac{\partial}{\partial u^a} - \sum_i D_i \frac{\partial}{\partial u_i^a} + \sum_{i \le j} D_i D_j \frac{\partial}{\partial u_{ij}^a} + \dots .$$

Consider an evolutionary vector field

$$X_\alpha = \alpha^a \frac{\partial}{\partial u^a} + \sum_i (D_i \alpha^a) \frac{\partial}{\partial u_i^a} + \sum_{i \le j} (D_i D_j \alpha^a) \frac{\partial}{\partial u_{ij}^a} + \dots \quad \alpha^a = \alpha^a(x, u, u_i, \dots). \tag{1}$$

Variation of the functional S under this infinitesimal transformation with operator X_α is

$$\delta S = \int_D X_\alpha L d^{m+1} x.$$

X_α is a variational (Noether) symmetry if

$$X_\alpha L = D_i M_i, \quad M_i = M_i(x, u, u_j, \dots), \quad i = 1, \dots, m+1, \tag{2}$$

The Noether identity

$$X_\alpha = \alpha^a E^a + D_i R_{\alpha i}, \quad R_{\alpha i} = \alpha^a \frac{\partial}{\partial u_i^a} + \left\{ \sum_{k \ge i} (D_k \alpha^a) - \alpha^a \sum_{k \le i} D_k \right\} \frac{\partial}{\partial u_{ik}^a} + \dots$$

in application to (2) we will obtain

$$D_i(M_i - R_{\alpha i} L) = \alpha^a \omega^a \equiv 0 \tag{3}$$

on the solution manifold ($\omega = 0$, $D_i \omega = 0, \dots$).

Thus, any 1-parameter variational symmetry transformation X_α (1) leads to a conservation law (3).

158 Nonlinear Klein-Gordon equation

$$u_{tt} - u_{x_1x_1} - \cdots - u_{x_dx_d} = F(u)$$

The equation is not integrable at $d > 1$ for any nonlinear F. At $d = 1$ the integrable nonlinear cases are exhausted, up to the point transforms, by three equations [2]:

- $u_{xy} = e^u$ the Liouville equation;
- $u_{xy} = \sin u$ the sine-Gordon equation;
- $u_{xy} = e^{2u} - e^{-u}$ the Tzitzeica equation.

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159 Nonlinear Schrödinger equation

$$u_t = u_{xx} + 2u^2v, \quad -v_t = v_{xx} + 2v^2u. \tag{1}$$

Aliases: Zakharov–Shabat, Ablowitz–Kaup–Newell–Segur system

➢ Third order symmetry:

$$u_{t_3} = u_{xxx} + 6uvu_x, \quad v_{t_3} = v_{xxx} + 6uvv_x. \tag{2}$$

➢ Bäcklund–Schlesinger transformation:

$$u_1 = u_{xx} - u_x^2/u + u^2v, \quad v_1 = 1/u. \tag{3}$$

The iterations of this mapping are governed, under the change $u = e^q$, $v = e^{-q_{-1}}$, by the Toda lattice

$$q_{xx} = e^{q_1 - q} - e^{q - q_{-1}}. \tag{4}$$

➢ Chain of Bäcklund–Darboux transformations:

$$u_{n,x} = u_{n+1} + \alpha_n u_n + u_n^2 v_{n+1}, \quad -v_{n,x} = v_{n-1} + \alpha_{n-1} v_n + u_{n-1} v_n^2 \tag{5}$$

$$\{v_m, u_n\} = \delta_{m,n+1}, \quad H = \sum\Big(u_n v_n + \alpha_n u_n v_{n+1} + \frac{1}{2}u_n^2 v_{n+1}^2\Big)$$

where α_n are arbitrary parameters. A generic BT for the NLS equation is decomposed as a sequence of elementary transformations of the form (3), (5) and their inverses.

➢ Permutability of the transformations (3) and (5) gives rise to 5-point equations of discrete Toda type

$$e^{q_{1,-i} - q} - e^{q - q_{-1,i}} + e^{q_i - q} - e^{q - q_{-i}} + \alpha^{(i)} - \alpha^{(i)}_{-i} = 0. \tag{6}$$

➢ Nonlinear superposition principle:

$$\tilde u_n = u_n - \frac{(\alpha_{n+1} - \alpha_n)u_{n-1}}{1 - u_{n-1}v_{n+1}}, \quad \tilde v_n = v_n + \frac{(\alpha_{n+1} - \alpha_n)v_{n+1}}{1 - u_{n-1}v_{n+1}} \tag{7}$$

➢ Zero curvature representation $U_t = V_x + [V, U]$, $W_x = U_1 W - WU$:

$$U = \begin{pmatrix} \lambda & -v \\ u & -\lambda \end{pmatrix}, \quad V = -2\lambda U + \begin{pmatrix} -uv & v_x \\ u_x & uv \end{pmatrix},$$
$$W_n = \begin{pmatrix} 1 & -v_{n+1} \\ u_n & -2\lambda - u_n v_{n+1} - \beta_n \end{pmatrix}$$

➢ Recursion operator:

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_n} = R^n \begin{pmatrix} u \\ v \end{pmatrix}, \qquad R = \begin{pmatrix} D_x + 2uD_x^{-1}v & 2uD_x^{-1}u \\ -2vD_x^{-1}v & -D_x - 2vD_x^{-1}u \end{pmatrix}$$

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160 Nonlinear Schrödinger equation, matrix

$$u_t = u_{xx} + 2uvu, \quad -v_t = v_{xx} + 2vuv, \quad u \in \mathrm{Mat}_{M,N}(\mathbb{C}), \quad v \in \mathrm{Mat}_{N,M}(\mathbb{C})$$

➢ Bäcklund transformation:

$$u_{n,x} = u_{n+1} + \beta_n u_n + u_n v_{n+1} u_n, \quad -v_{n,x} = v_{n-1} + \beta_{n-1} v_n + v_n u_{n-1} v_n$$

Third order symmetry:

$$u_t = u_{xxx} + 3u_x vu + 3uvu_x, \quad v_t = v_{xxx} + 3v_x uv + 3vuv_x.$$

➢ Zero curvature representation

$$U = \begin{pmatrix} -\lambda M I_N & -v \\ u & \lambda N I_M \end{pmatrix}, \quad V = \lambda(M+N)U + \begin{pmatrix} -vu & v_x \\ u_x & uv \end{pmatrix}$$
$$W_n = \begin{pmatrix} I_N & -v_{n+1} \\ u_n & \lambda(M+N)I_M - \beta_n I_M - u_n v_{n+1} \end{pmatrix}.$$

The $M \times M$ matrices

$$U = -2uv, \quad W = 2uv_x - 2u_x v$$

satisfy the matrix KP equation

$$4U_{t_3} = U_{xxx} - 3(U_x U + UU_x - W_t + [W,U]), \quad W_x = U_t,$$

and the matrices

$$F_n = -u_n v_{n+1} - \beta_n I_M, \quad P_n = u_n v_{n+1,x} - u_{n,x} v_{n+1} + u_n v_{n+1} u_n v_{n+1} - \beta_n^2 I_M$$

satisfy the two-dimensional matrix dressing chain

$$F_{n+1,x} + F_{n,x} = F_{n+1}^2 - F_n^2 + P_{n+1} - P_n, \quad P_{n,x} = F_{n,t} + [P_n, F_n].$$

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161 Nonlinear Schrödinger equation, multidimensional

$$i\psi_t = \Delta\psi + |\psi|^{2\sigma}\psi, \quad \Delta = \nabla^2 = \partial^2_{x_1} + \cdots + \partial^2_{x_n} \tag{1}$$

➢ This equation arises in a number of physical problems, such as plasma physics and nonlinear optics [1]. Not integrable. Some particular solutions describing the weak collapse and governed by ODE of Painlevé type were studied in [2, 3, 4].

The conserved densities:

$$|\psi|^2, \qquad \frac{i}{2}(\psi\nabla\psi^* - \psi^*\nabla\psi), \qquad \frac{1}{2}|\nabla\psi|^2 - \frac{1}{\sigma+1}|\psi|^{2\sigma+2}.$$

References

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162 Nonlinear Schrödinger equation, Jordan

$$u_t = u_{xx} + 2\{uvu\}, \quad -v_t = v_{xx} + 2\{vuv\}, \quad u \in V^+, \quad v \in V^-$$

where $V = (V^+, V^-)$ is a Jordan pair.

This is the most general multifield version of NLS. The particular cases are:

- the matrix NLS system;
- the Manakov vectorial NLS system;
- the Kulish–Sklyanin vectorial NLS system.

➢ Third order symmetry:

$$u_t = u_{xxx} + 6\{uvu_x\}, \quad v_t = v_{xxx} + 6\{vuv_x\}.$$

➢ Bäcklund transformation::

$$u_{j,x} = u_{j+1} + \beta_j u_j + \{u_j v_{j+1} u_j\}, \quad -v_{j,x} = v_{j-1} + \beta_{j-1} v_j + \{v_j u_{j-1} v_j\}$$

➢ Zero curvature representation $U_t = V_x + [V, U]$ is given in terms of the structure Lie algebra of Jordan pair:

$$U = u - 2v + \lambda\sigma, \quad V = u_x + 2v_x + 2L(u, v) + \lambda U$$

➢ The differential substitution $v = -w_x - \{wuw\}$ (equivalent to the shift $v_{j+1} \to v_j$ in the chain of BT) brings to the modified Jordan NLS

$$u_t = u_{xx} - 2\{uw_xu\} - 2\{u\{wuw\}u\}, \quad -w_t = w_{xx} + 2\{wu_xw\} - 2\{w\{uwu\}w\}$$

which is the symmetry of the PLR-type hyperbolic system

$$u_{xy} = 2\{uwu_y\} - u, \quad w_{xy} = -2\{wuw_y\} - w.$$

References

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163 Nonlinear Schrödinger equation, vectorial

Author: V.E. Adler, 2007.02.05

$$u_t = u_{xx} + 4\langle u,v\rangle u - 2\langle u,u\rangle v, \quad -v_t = v_{xx} + 4\langle u,v\rangle v - 2\langle v,v\rangle u, \quad u,v \in \mathbb{C}^m. \tag{1}$$

Alias: Kulish–Sklyanin system

➢ Introduced in [1].

➢ Third order symmetry:

$$\begin{aligned} u_t &= u_{xxx} + 6\langle u,v\rangle u_x + 6\langle u_x,v\rangle u - 6\langle u,u_x\rangle v,\\ v_t &= v_{xxx} + 6\langle u,v\rangle v_x + 6\langle u,v_x\rangle v - 6\langle v,v_x\rangle u. \end{aligned} \tag{2}$$

➢ Bäcklund–Schlesinger transformation [2]:

$$u_1 = u_{xx} - 2\frac{\langle u,u_x\rangle}{\langle u,u\rangle}u_x + \frac{\langle u_x,u_x\rangle}{\langle u,u\rangle}u + 2\langle u,v\rangle u - \langle u,u\rangle v, \quad v_1 = \frac{1}{\langle u,u\rangle}u, \tag{3}$$

➢ Bäcklund–Darboux transformation:

$$u_x = u_i + \alpha^{(i)}u + 2\langle u,v_i\rangle u - \langle u,u\rangle v_i, \quad -v_{i,x} = v + \alpha^{(i)}v_i + 2\langle u,v_i\rangle v_i - \langle v_i,v_i\rangle u. \tag{4}$$

➢ Nonlinear superposition principle [3]:

$$u_j = u_i + \frac{(\alpha^{(i)}-\alpha^{(j)})(u - \langle u,u\rangle v_{ij})}{1 - 2\langle u,v_{ij}\rangle + \langle u,u\rangle\langle v_{ij},v_{ij}\rangle}, \quad v_j = v_i - \frac{(\alpha^{(i)}-\alpha^{(j)})(v_{ij} - \langle v_{ij},v_{ij}\rangle u)}{1 - 2\langle u,v_{ij}\rangle + \langle u,u\rangle\langle v_{ij},v_{ij}\rangle}. \tag{5}$$

➢ Zero curvature representation:

$$U = \begin{pmatrix} -\lambda & -2v^{\mathsf T} & 0\\ u & 0 & 2v\\ 0 & -u^{\mathsf T} & \lambda \end{pmatrix}, \quad V = \lambda U + \begin{pmatrix} -2v^{\mathsf T}u & 2v_x^{\mathsf T} & 0\\ u_x & 2uv^{\mathsf T} - 2vu^{\mathsf T} & -2v_x\\ 0 & -u_x^{\mathsf T} & 2u^{\mathsf T}v \end{pmatrix}$$

$$W^{(i)} = \begin{pmatrix} 1 & -2v_i^{\mathsf T} & -2v_i^{\mathsf T}v_i \\ u & (\lambda-\alpha^{(i)})I_n - 2uv_i^{\mathsf T} & 2(\lambda-\alpha^{(i)}-uv_i^{\mathsf T})v_i \\ -\frac12 u^{\mathsf T}u & u^{\mathsf T}(uv_i^{\mathsf T}-\lambda+\alpha^{(i)}) & (\lambda-\alpha^{(i)})^2-2(\lambda-\alpha^{(i)})u^{\mathsf T}v_i+u^{\mathsf T}uv_i^{\mathsf T}v_i \end{pmatrix}$$

➢ Kulish–Sklyanin hierarchy is a squared eigenfunction constraint for the Hirota–Ohta hierarchy:

Statement 1. *Equations (1)–(5) are consistent and, in virtue of these equations, the quantities*

$$U=-\langle u,u\rangle,\quad V=-\langle v,v\rangle,\quad W=4\langle u,v\rangle,\quad Q=4\langle u_x,v\rangle-4\langle u,v_x\rangle,$$
$$W^{(i)}=\alpha^{(i)}+2\langle u,v_i\rangle,\quad W^{(ij)}=\frac{\alpha^{(i)}-\alpha^{(j)}}{1-2\langle u,v_{ij}\rangle+\langle u,u\rangle\langle v_{ij},v_{ij}\rangle}$$

satisfy the equations of the Hirota–Ohta hierarchy.

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164 Nonlinear Schrödinger equation, derivative

$$u_t = u_{xx} + 2(u^2v)_x, \quad v_t = -v_{xx} + 2(uv^2)_x$$

Alias: Kaup–Newell system, DNLS-I; Namesakes: Chen–Lee–Liu system (DNLS-II), Gerdjikov–Ivanov equation (DNLS-III)

➢ Master-symmetry:

$$u_\tau = (xu_x + 2xu^2v + cu)_x, \quad v_\tau = (-xv_x + 2xuv^2 + (c-1)v)_x$$

➢ Higher symmetry:

$$u_{t_3} = (u_{xx} + 6uu_xv + 6u^3v^2)_x, \quad v_{t_3} = (v_{xx} - 6uvv_x + 6u^2v^3)_x$$

➢ Bäcklund transformation:

$$u_{n,x} = u_n^2(u_{n+1} - u_{n-1}), \quad u_{n,t} = u_n^2(u_{n+1}^2(u_{n+2} + u_n) - u_{n-1}^2(u_n - u_{n-2})), \quad u = u_n, \quad v = u_{n-1}.$$

➢ Zero curvature representation

$$U = 2\lambda\begin{pmatrix}\lambda & u\\ -v & -\lambda\end{pmatrix}, \quad V = (4\lambda^2 + 2uv)U + 2\lambda\begin{pmatrix}0 & u_x\\ v_x & 0\end{pmatrix}, \quad L = \begin{pmatrix}2\lambda/u & 1\\ -1 & 0\end{pmatrix}.$$

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165 Nonlinear Schrödinger equation, Jordan derivative

$$u_t = u_{xx} + 2\{uvu\}_x, \quad v_t = -v_{xx} + 2\{vuv\}_x, \quad u \in V^+, \quad v \in V^-$$

where $V = (V^+, V^-)$ is a Jordan pair.

This is the most general multifield version of the DNLS-I equation.

Third order symmetry:

$$u_{t_3} = (u_{xx} + 6\{uvu_x\} + 6\{u\{vuv\}u\})_x, \quad v_{t_3} = (v_{xx} - 6\{vuv_x\} + 6\{v\{uvu\}v\})_x.$$

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166 Nonlinear Schrödinger equation, matrix derivative

$$u_t = u_{xx} + 2(uv^\intercal u)_x, \quad v_t = -v_{xx} + 2(vu^\intercal v)_x, \quad u, v \in \mathrm{Mat}_{m,n}(\mathbb{C})$$

This and some others matrix versions of DNLS-type equations were studied in [1, 2, 3].

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167 Nonlinear Schrödinger equation, vectorial derivative

$$u_t = u_{xx} + 2(\langle u, v\rangle u)_x, \quad v_t = -v_{xx} + 2(\langle u, v\rangle v)_x, \quad u, v \in \mathbb{C}^n$$

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168 Nonlinear Schrödinger type systems, classification

Author: V.E. Adler, 09.01.2009

1. Introduction

The classification problem of ***NLS type*** integrable systems

$$w_t = A(w)w_2 + F(w, w_1), \quad w = (u, v)^{\mathsf{T}}, \quad F = (f, g)^{\mathsf{T}}, \quad \det A \neq 0 \tag{1}$$

within the symmetry approach was considered in the papers [1] where the necessary conditions were obtained for the existence of the higher order conservations laws. These conditions separate out the nonintegrable cases as well as the linearizable systems of Burgers type which possess higher symmetries, but do not possess the higher conservations laws. In particular, it turned out that all systems which satisfy these conditions can be brought to the form

$$u_t = u_2 + f(u, v, u_1, v_1), \quad -v_t = u_2 + g(u, v, u_1, v_1) \tag{2}$$

by use of a differential substitution. Further classification was done modulo point changes plus so-called symmetric transformations. In the final form this problem was solved in the paper [2], see also [3, 4]. Several systems appeared to be new and their integrability was justified either by establishing a differential substitution bringing to a known integrable system either by construction of zero curvature representation. Depending on the order of the auxiliary linear problem, 2 and 3 correspondingly, the list is divided into the NLS-type systems (a–p) and Boussinesq-type systems (q–u6).

2. Extension of the module of the point transformations

The point changes acting on the whole set of the systems (2) are generated by the transformations

$$(x,t) \to (ax+bt+c, a^2t+d), \qquad (u,v) \to (\phi(u), \psi(v)), \qquad (x,t,u,v) \to (-x,-t,v,u). \tag{3}$$

The numerous subclass consists of the systems which are invariant with respect to some one-parametric group of transformations $(u,v) \to (\phi(\alpha,u), \psi(\alpha,v))$. One can assume, without loss of generality that this subgroup consists of the shifts $(u,v) \to (u+\alpha, v)$ or $(u,v) \to (u+\alpha, v-\alpha)$. The corresponding system is of the special form

$$u_t = u_2 + f(\varepsilon u + v, u_1, v_1), \quad -v_t = u_2 + g(\varepsilon u + v, u_1, v_1), \quad \varepsilon = 0, 1. \tag{4}$$

Such a system admits the differential substitution of the form $\tilde{u} = U(\varepsilon u + v, u_1)$, $\tilde{v} = V(\varepsilon u + v)$, which generically leads beyond the class (4). However, there is an important case when a composition of such systems preserves the form of the system.

A ***symmetric system*** is the system of the form (4) which is invariant with respect to the involution $(x,t,u,v) \to (-x,-t,v,u)$:

$$u_t = u_2 + f(u+v, u_1, v_1), \quad -v_t = u_2 + f(u+v, -v_1, -u_1). \tag{5}$$

The following properties are valid.

Theorem 1. *Let the system (5) possesses a conservation law with the density $\rho = p'(u+v)u_1 + q(u+v)$, $p' \neq 0$. Then the* ***symmetric transformation***

$$\tilde{u} + \tilde{v} = p(u+v), \quad \tilde{u}_1 = p'(u+v)u_1 + q(u+v) \tag{6}$$

maps it to another symmetric system. Transformations of this form define the equivalence relation on the set of the systems (5) and preserve the integrability property, that is if the original system possesses the higher symmetries and conservation laws then so its transform does.

The following example demonstrates that the use of symmetric changes allow to reduce essentially the list of integrable systems.

Example 2. Let us consider the system

$$u_t = u_2 + 2auvu_1 + bu^2v_1 + \frac{1}{2}b(a-b)u^3v^2 + cu^2v, \quad -v_t = v_2 - 2auvv_1 - bv^2u_1 + \frac{1}{2}b(a-b)u^2v^3 + cuv^2$$

which includes, for instance, the complexified Gerdjikov–Ivanov system as a particular case. The chage $u \to \exp(u)$, $v \to \exp(v)$ brings it to the form (5) with $f = u_1^2 + (2au_1 + bv_1 + c)e^{u+v} + \frac{1}{2}b(a-b)e^{2u+2v}$. It is not difficult to establish, by use of the density $\rho = u_1 + \beta e^{u+v}$, the symmetric equivalence with the following systems:

1) at $b = 2a, c = 0$ with the linear system $(a = b = c = 0)$;
2) at $b = 2a, c \neq 0$ with NLS $(a = b = 0, c = 1)$;
3) at $b \neq 2a$ with DNLS $(a = b = 1, c = 0)$.

3. The list of integrable systems

Theorem 3 ([2]). *The systems (2) possessing an infinite set of higher symmetries and conservation laws are reducible to the systems of the following list, up to the transformations (3) and symmetric transformations (6).*

Remark. In some instances it is convenient to include the equations which are equivalent modulo the aforementioned transformations. Such systems are denoted by the same letters with primes. The upper- and lower-case marked system are related via potentiation. The other changes are described in the next section.

	$u_t = u_2 + u_1^2 + v_1, \quad -v_t = v_2 - 2u_1v_1;$	(a)
Kaup–Broer	$u_t = u_2 + (u^2 + v)_x, \quad -v_t = v_2 - 2(uv)_x;$	(A)
NLS	$u_t = u_2 + u^2v, \quad -v_t = v_2 + v^2u;$	(b)
Kaup	$u_t = u_2 + (u+v)u_1, \quad -v_t = v_2 - (u+v)v_1;$	(c)
	$u_t = u_2 + u_1^2v_1 - 4v_1, \quad -v_t = v_2 - u_1v_1^2 + 4u_1;$	(d)
	$u_t = u_2 + (u^2v - 4v)_x, \quad -v_t = v_2 - (uv^2 - 4u)_x;$	(D)

$$u_t = u_2 - \frac{u_1^2 v_1}{(u+v)^2} - \frac{2u_1^2}{u+v}, \quad -v_t = v_2 + \frac{u_1 v_1^2}{(u+v)^2} - \frac{2v_1^2}{u+v}; \tag{d$'$}$$

$$\begin{cases} u_t = u_2 + \operatorname{sech}^2(u+v)u_1^2 v_1 - 2\tanh(u+v)u_1^2, \\ -v_t = v_2 - \operatorname{sech}^2(u+v)u_1 v_1^2 - 2\tanh(u+v)v_1^2; \end{cases} \tag{d$''$}$$

$$u_t = u_2 - 2\tanh(u+v)(u_1^2 - 4), \quad -v_t = v_2 - 2\tanh(u+v)(v_1^2 - 4); \tag{e}$$

$$\begin{cases} u_t = u_2 - \dfrac{2u_1^2}{u+v} - \dfrac{8(1+uv)u_1 + 4(1-u^2)v_1}{(u+v)^2}, \\ -v_t = v_2 - \dfrac{2v_1^2}{u+v} + \dfrac{8(1+uv)v_1 + 4(1-v^2)u_1}{(u+v)^2}; \end{cases} \tag{f}$$

$$u_t = u_2 + u_1^2 v_1, \quad -v_t = v_2 - u_1 v_1^2 - u_1; \tag{g}$$

$$u_t = u_2 + (u^2 v)_x, \quad -v_t = v_2 - (uv^2 + u)_x; \tag{G}$$

$$u_t = u_2 + u_1^2 - 2u_1 v_1, \quad -v_t = v_2 - v_1^2 - 2u_1 v_1; \tag{h}$$

Levi

$$u_t = u_2 + (u^2 - 2uv)_x, \quad -v_t = v_2 - (v^2 - 2uv)_x; \tag{H}$$

Heisenberg

$$u_t = u_2 - \frac{2u_1^2}{u+v}, \quad -v_t = v_2 - \frac{2v_1^2}{u+v}; \tag{h$'$}$$

$$u_t = u_2 - 2\tanh(u+v)u_1^2, \quad -v_t = v_2 - 2\tanh(u+v)v_1^2; \tag{h$''$}$$

$$u_t = u_2 + u_1^2 v_1, \quad -v_t = v_2 - u_1 v_1^2; \tag{i}$$

DNLS

$$u_t = u_2 + (u^2 v)_x, \quad -v_t = v_2 - (uv^2)_x; \tag{I}$$

$$u_t = u_2 + \exp(u+v)u_1^2 v_1 + u_1^2, \quad -v_t = v_2 - \exp(u+v)u_1 v_1^2 + v_1^2; \tag{i$'$}$$

$$u_t = u_2 - \frac{2(u_1^2+1)}{u+v}, \quad -v_t = v_2 - \frac{2(v_1^2+1)}{u+v}; \tag{j}$$

$$\begin{cases} u_t = u_2 - \dfrac{2u_1^2}{u+v} - \dfrac{4((u-v)u_1 + uv_1)}{(u+v)^2}, \\ -v_t = v_2 - \dfrac{2v_1^2}{u+v} + \dfrac{4((u-v)v_1 - u_1 v)}{(u+v)^2}; \end{cases} \tag{k}$$

$$\begin{cases} u_t = u_2 + R(y)u_1^2 v_1 + R'(y)u_1^2 - \dfrac{2}{3}(R''(y) - 2c)u_1 + \dfrac{1}{3}R'''(y), \\ -v_t = v_2 - R(y)u_1 v_1^2 + R'(y)v_1^2 + \dfrac{2}{3}(R''(y) - 2c)v_1 + \dfrac{1}{3}R'''(y), \end{cases} \tag{l}$$

where $y = y(u+v), \quad y' = R(y) = ay^4 + by^3 + cy^2 + dy + e \neq 0;$

$$\begin{cases} u_t = u_2 - \dfrac{2u_1^2}{u+v} - \dfrac{4(P(u,v)u_1 + R(u)v_1)}{(u+v)^2}, \\ -v_t = v_2 - \dfrac{2v_1^2}{u+v} + \dfrac{4(P(u,v)v_1 + R(-v)v_1)}{(u+v)^2}, \end{cases} \tag{m}$$

where $P(u,v) = 2au^2v^2 + buv(v-u) - 2cuv + d(u-v) + 2e,$

$R(y) = ay^4 + by^3 + cy^2 + dy + e;$

Landau–Lifshits

$$\begin{cases} u_t = u_2 - \dfrac{2(u_1^2 + R(u))}{u+v} + \dfrac{R'(u)}{2}, \\ -v_t = v_2 - \dfrac{2(v_1^2 + R(-v))}{u+v} - \dfrac{R'(-v)}{2}, \end{cases} \tag{n}$$

where $R(y) = ay^4 + by^3 + cy^2 + dy + e;$

$$\begin{cases} u_t = u_2 + e^{\phi}(u_1^2 + 1)v_1 + \phi_u u_1^2 + 2(y(u+v) + y(u-v))u_1, \\ -v_t = v_2 - e^{\phi}(v_1^2 + 1)u_1 + \phi_v v_1^2 - 2(y(u+v) + y(u-v))v_1, \end{cases} \tag{o}$$

where $e^{\phi} = y(u+v) - y(u-v),$

$(y')^2 = -4y^4 + ay^3 + by^2 + cy + d;$

$$u_t = u_2 + (e^{\phi} v_1 + \phi_u)(u_1^2 + 1), \quad -v_t = v_2 - (e^{\phi} u_1 - \phi_v)(v_1^2 + 1), \tag{p}$$

$$\text{where} \quad e^{\phi} = y(u+v) - y(u-v),$$
$$(y')^2 = -y^4 + ay^3 + by^2 + cy + d;$$

Boussinesq

$$u_t = u_2 + v_1, \quad -v_t = v_2 - u_1^2; \tag{q}$$

$$u_t = u_2 + v_1, \quad -v_t = v_2 - (u^2)_x; \tag{Q}$$

$$u_t = u_2 + (u+v)^2, \quad -v_t = v_2 + (u+v)^2; \tag{r}$$

$$u_t = u_2 + (u+v)v_1 - \frac{1}{6}(u+v)^3, \quad -v_t = v_2 - (u+v)u_1 - \frac{1}{6}(u+v)^3; \tag{s}$$

$$u_t = u_2 + v_1, \quad -v_t = v_2 - u_1^2 - (v + \frac{1}{2}u^2)u_1; \tag{t}$$

$$u_t = u_2 + v_1^2, \quad -v_t = v_2 + u_1^2; \tag{u1}$$

$$u_t = u_2 + 2vv_1, \quad -v_t = v_2 + 2uu_1; \tag{U1}$$

(In the systems (u2)–(u6) the notation $\omega = \exp(\frac{2\pi i}{3})$, $E = e^{u+v}$, $E_1 = e^{\omega u + \omega^2 v}$, $E_2 = e^{\omega^2 u + \omega v}$) is used.

$$u_t = u_2 + v_1^2 + bE - 2cE^{-2}, \quad -v_t = v_2 + v_1^2 + bE - 2cE^{-2}; \tag{u2}$$

$$\begin{cases} u_t = u_2 + v_1^2 - (aE^{-1} + \omega a_1 E_1^{-1} + \omega^2 a_2 E_2^{-1})v_1, \\ -v_t = v_2 + u_1^2 + (aE^{-1} + \omega^2 a_1 E_1^{-1} + \omega a_2 E_2^{-1})u_1; \end{cases} \tag{u3}$$

$$\begin{cases} u_t = u_2 + v_1^2 - 2cE^{-2} - 2\omega^2 c_1 E_1^{-2} - 2\omega c_2 E_2^{-2}, \\ -v_t = v_2 + u_1^2 - 2cE^{-2} - 2\omega c_1 E_1^{-2} - 2\omega^2 c_2 E_2^{-2}; \end{cases} \tag{u4}$$

$$u_t = u_2 + v_1^2 + bE + \omega^2 b_1 E_1 + \omega b_2 E_2, \quad -v_t = v_2 + u_1^2 + bE + \omega b_1 E_1 + \omega^2 b_2 E_2; \tag{u5}$$

$$\begin{cases} u_t = u_2 + v_1^2 - (aE^{-1} + \omega a_1 E_1^{-1} + \omega^2 a_2 E_1^{-1})v_1 \\ \qquad - \frac{1}{6}(a_1 a_2 E + \omega^2 a a_2 E_1 + \omega a a_1 E_2 + a^2 E^{-2} + \omega^2 a_1^2 E_1^{-2} + \omega a_2^2 E_2^{-2}), \\ -v_t = v_2 + u_1^2 + (aE^{-1} + \omega^2 a_1 E_1^{-1} + \omega a_2 E_2^{-1})u_1 \\ \qquad - \frac{1}{6}(a_1 a_2 E + \omega a a_2 E_1 + \omega^2 a a_1 E_2 + a^2 E^{-2} + \omega a_1^2 E_1^{-2} + \omega^2 a_2^2 E_2^{-2}); \end{cases} \tag{u6}$$

$$u_t = u_2 - \frac{(u_1 + 2v_1)u_1}{2(u+v)} + a(u+v), \quad -v_t = v_2 - \frac{(2u_1 + v_1)v_1}{2(u+v)} + b(u+v); \tag{v}$$

$$u_t = u_2 + (u^2 + v^{-1})_x, \quad -v_t = v_2 - 2(uv)_x - 1. \tag{w}$$

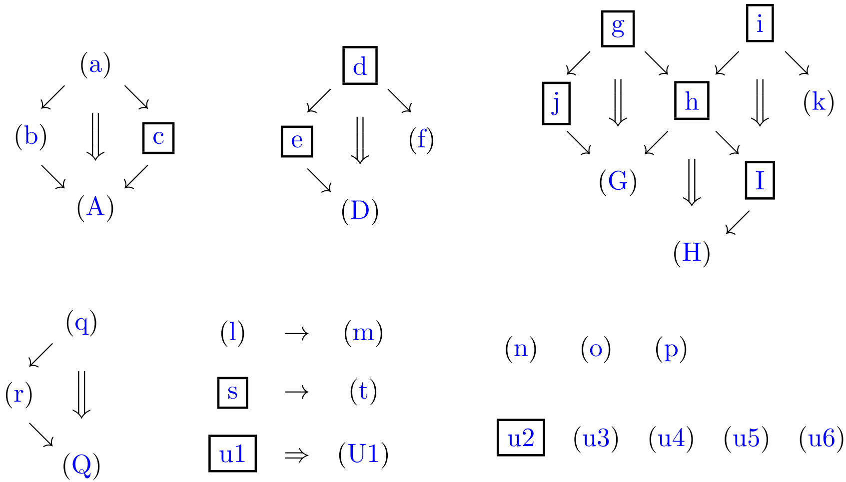
4. Substitutions

The systems (v), (w) can be brought to the linear and reducible systems, respectively:

$$(\text{v}) \to \ (u_t = u_2 + v_1 + \frac{a-b}{2}u, \quad -v_t = v_2 - 2bu_1 + \frac{a-b}{2}v) \ : \quad \tilde{u} = 2(u+v)^{1/2}, \quad \tilde{v} = -2v_1(u+v)^{-1/2},$$

$$(\text{w}) \to \ (u_t = u_2 + 1/v, \quad -v_t = v_2) \ : \quad \tilde{u} = u_1/u, \quad \tilde{v} = uv.$$

The other systems are related by the following changes (the symmetric systems are in boxes, the double arrows denote potentiation $\tilde{u} = u_1$, $\tilde{v} = v_1$, and, as usually, in the substitution marked $A \to B$ the tilded variables correspond to equation B):



(a) → (b)	$\tilde{u} = e^u,$	$\tilde{v} = e^{-u} v_1$
(b) → (A)	$\tilde{u} = u_1/u,$	$\tilde{v} = uv$
(c) → (A)	$\tilde{u} = (u+v)/2,$	$\tilde{v} = -v_1$
(a) → (c)	$\tilde{u} = 2u_1 + v,$	$\tilde{v} = -v$
(d) → (e)	$\tilde{u} = \operatorname{atanh}(u_1/2) - v,$	$\tilde{v} = v$
(e) → (D)	$\tilde{u} = 2\tanh(u+v),$	$\tilde{v} = v_1$
(d″) → (f)	$\tilde{u} = \tanh(u+v),$	$\tilde{v} = -\tanh(u+v) - 2/v_1$
(g) → (j)	$\tilde{u} = 2/u_1,$	$\tilde{v} = v$
(j) → (G)	$\tilde{u} = 2/(u+v),$	$\tilde{v} = v_1$
(g) → (h″)	$\tilde{u} = -iu/2,$	$\tilde{v} = iu/2 + \operatorname{atanh}(-iv_1)$
(h″) → (G)	$\tilde{u} = 2iu_1,$	$\tilde{v} = i\tanh(u+v)$
(h′) → (I)	$\tilde{u} = 2/(u+v),$	$\tilde{v} = v_1$
(I) → (H)	$\tilde{u} = -uv/2,$	$\tilde{v} = -uv/2 - v_1/v$
(i) → (h′)	$\tilde{u} = 2/u_1 - v_1,$	$\tilde{v} = v$
(i′) → (k)	$\tilde{u} = e^{u+v},$	$\tilde{v} = -2/v_1 - e^{u+v}$
(l) → (m)	$\tilde{u} = y(u+v),$	$\tilde{v} = -2/v_1 - y(u+v)$
(q) → (r)	$\tilde{u} = u_1/2 + v/4,$	$\tilde{v} = -v/4$
(r) → (Q)	$\tilde{u} = 2(u+v),$	$\tilde{v} = -4v_1$
(s) → (t)	$\tilde{u} = -(u+v),$	$\tilde{v} = 2v_1 - (u+v)^2/2$

5. Necessary integrability conditions

Statement 4. *If the system (1) possesses the conservation law with the density $\rho(w, w_1, \dots, w_n)$ of nonzero order then*

$$\operatorname{tr} A = 0, \quad \operatorname{tr}(A^{-1}F_{w_1}) \in \operatorname{Im} D_x, \quad D_t((\det A)^{-1/4}) \in \operatorname{Im} D_x \tag{7}$$

where $F_{w_1} = \begin{pmatrix} f_{u_1} & f_{v_1} \\ g_{u_1} & g_{v_1} \end{pmatrix}$ denotes the Jacobi matrix. Moreover, the density ρ is a polynomial in w_n, and its degree does not exceed 2.

The Statement 4 allows to make the change which simplifies the system to the form (2). The further integrability conditions are computed for the systems which are already in this form. These conditions mean that the equations

$$D_t(\rho_k) = D_x(\sigma_k), \quad \omega_k = D_x(\phi_k), \quad k = 0, 1, 2, \dots \tag{8}$$

must be solvable with respect to σ_k, ϕ_k, as functions on u, v and their x-derivatives, where ρ_k and ω_k are determined through the r.h.s. of the systems and $\sigma_0, \dots, \sigma_{k-1}$, $\phi_0, \dots, \phi_{k-1}$ found previously, accordingly to the formulae from the following statement. It turns out that the complete description of integrable cases requires only four first conditions.

Statement 5. *If the system (2) possesses the conservation laws and symmetries of the higher enough order, then the conditions (8) are fulfilled, where*

$$\begin{aligned}
\rho_0 &= \frac{1}{2} f_{u_1} - \frac{1}{2} g_{v_1}, \\
\rho_1 &= \sigma_0 - \frac{1}{4} f_{u_1}^2 - \frac{1}{4} g_{v_1}^2 - f_{v_1} g_{u_1} + f_u + g_v, \\
\rho_2 &= \sigma_1, \\
\rho_3 &= \sigma_2 + \frac{1}{2}\rho_1^2 + \frac{1}{2}\omega_1^2 - \omega_0(\omega_2 - D_t(\phi_1)) - 4 f_v g_u + f_{v_1} D_t(g_{u_1}) - D_t(f_{v_1}) g_{u_1} \\
&\quad + D_t(f_u - g_v) - f_{v_1}^2 g_{u_1}^2 + 2 f_{v_1} g_{u_1}(f_u + g_v) - 2 D_x(f_{v_1}) D_x(g_{u_1}) \\
&\quad + \frac{1}{2}(D_x(\omega_0))^2 + \frac{1}{2}(D_x(\rho_0))^2 + \rho_0 (D_x(f_{v_1}) g_{u_1} - f_{v_1} D_x(g_{u_1}))
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2g_u D_x(f_{v_1}) + 2f_v D_x(g_{u_1}) - f_u D_x(f_{u_1}) - g_v D_x(g_{v_1}),\\
\omega_0 &= \frac{1}{2} f_{u_1} + \frac{1}{2} g_{v_1},\\
\omega_1 &= D_t(\phi_0) - \phi_0\rho_0 - f_{v_1} g_{u_1} + f_u - g_v,\\
\omega_2 &= D_t(\phi_1) + 2\omega_0 f_{v_1} g_{u_1} - 2f_{v_1} g_u - 2f_v g_{u_1},\\
\omega_3 &= D_t(\phi_2) + \rho_1\omega_1 - \rho_0(\omega_2 - D_t(\phi_1)) + D_t(f_u + g_v) + \omega_0(D_x(f_{v_1}) g_{u_1} - f_{v_1} D_x(g_{u_1}))\\
&\quad + D_x(\omega_0) D_x(\rho_0) - f_u D_x(f_{u_1}) + g_v D_x(g_{v_1}) - 2D_x(f_{v_1}) g_u + 2f_v D_x(g_{u_1}).
\end{aligned}
$$

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169 N-wave equation, twodimensional

$$u_{ij,t} = \frac{\omega_i - \omega_j}{\alpha_i - \alpha_j} u_{ij,x} + \frac{\alpha_j \omega_i - \alpha_i \omega_j}{\alpha_i - \alpha_j} u_{ij,y} + \sum_{k=1, k \neq i,j}^{N} \left(\frac{\omega_i - \omega_k}{\alpha_i - \alpha_k} + \frac{\omega_k - \omega_j}{\alpha_k - \alpha_j} \right) u_{ik} u_{kj} \qquad (1)$$

where $i, j = 1, \ldots, N$, $i \neq j$, $\alpha_i \neq \alpha_j$, $\omega_i \neq \omega_j$.

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170 Orthogonal lattice

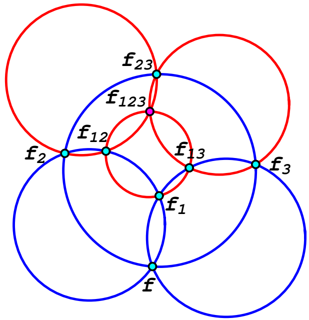
Author: V.E. Adler, Last. mod.: 1.12.2008

Planar lattices admit numerous important special cases. One of the possible reductions is the following.

Definition 1. The mapping $f:\mathbb{Z}^M\to\mathbb{R}^d$, $d>1$ is called M-dimensional ***orthogonal lattice*** (= ***circular lattice*** = ***discrete orthogonal net***) is the image of any unit square in $\mathbb{Z}^M$ is a planar inscribed quadrangle.

Obviously, if $d>2$ and three 2-dimensional circular lattices are given as the initial data on the coordinate planes then the whole lattice is constructed by use of the planarity condition only, just as in the case of generic planar lattices. The fact, that this construction preserves the property of the faces to be inscribed is guaranteed by Miquel theorem.

Theorem 2 (Miquel)**.** *Let three circles C^{ij} and seven points f, f_i, $f_{ij}=f_{ji}$, $1\le i,j\le 3$, $i\ne j$ be given, such that f, f_i, f_j, $f_{ij}\in C^{ij}$. Then three circles C^{ij}_k through the points f_k, f_{ki}, f_{kj} meet in a point: $f_{123}=C^{12}_3\cap C^{13}_2\cap C^{23}_1$.*



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171 Painlevé property

Definition 1. An ODE in complex plane possesses the Painlevé property if the position of any essential singularity in its solution does not depend on the initial data. In other words, all *movable* singularities, if any, are poles.

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172 Painlevé test

The Ablowitz–Ramani–Segur conjecture [1] states that a nonlinear PDE is solvable by the ISTM only if its every ODE reduction possesses the Painlevé property.

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173 Painlevé equation

The works [1, 2] were devoted to the classification of second order ODE $y'' = f(z, y, y')$ with the r.h.s. rational in y, y' and analytic in z, which satisfy the Painlevé property. There exist 50 types of such equations, up to the changes

$$\tilde{z} = f(z), \quad \tilde{y} = \frac{a(z)y + b(z)}{c(z)y + d(z)}$$

where a, b, c, d, f are analytic functions [3]. The most part is solved in the elementary or elliptic functions, the others can be brought to six irreducible cases known as ***Painlevé equations*** P_1–P_6. The general solutions of the latter are special functions called ***Painlevé transcendents***. Selfsimilar solutions of nonlinear integrable PDEs and lattices can be often expressed through these functions or their higher analogs. Some special solution classes of P_2–P_6 equations (characterized by certain values of parameters and initial data) are expressed through elementary functions or through hypergeometric type functions.

The main tool in the theory of Painlevé equation is the isomonodromy deformations method, based on the representation of these equation as the compatibility conditions of certain linear equations. This method allows to find the asymptotic of Painlevé transcendents and their dependence on the initial data.

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174 Painlevé equation P_1

$$u'' = 6u^2 + z \qquad (P_1)$$

Representation by entire functions: $u = -(\log f)''$,

$$f f^{\mathrm{IV}} - 4f' f''' + 3(f'')^2 + z f^2 = 0.$$

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175 Painlevé equation P_2

$$u'' = 2u^3 + zu + \alpha \qquad (\mathrm{P}_2)$$

Representation by entire functions: $u = g/f$,

$$ff'' - (f')^2 + g^2 = 0, \quad (f'g - fg')^2 = g^4 + zf^2g^2 + (2\alpha g + f')f^3.$$

Bäcklund transformations

$$\hat{u} = u \pm \frac{2a \pm 1}{2u' \pm 2u^2 \pm z}, \quad \hat{a} = \pm 1 - a$$

allow to generate solutions for all values of the parameter $a + 2n$, $-a + 2n + 1$, $n \in \mathbb{Z}$.

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176 Painlevé equation P_3

$$u'' = \frac{(u')^2}{u} - \frac{u'}{z} + \frac{1}{z}(\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u} \tag{P_3}$$

Representation by entire functions: $u = g/f$,

$$ff'' - (f')^2 = -\gamma e^{2z} f^2 - \alpha e^z fg, \quad gg'' - (g')^2 = \delta e^{2z} f^2 + \beta e^z fg.$$

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177 Painlevé equation P_4

$$u'' = \frac{(u')^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2(z^2 - \alpha)u + \frac{\beta}{u} \tag{P_4}$$

Representation by entire functions: $u = g/f$,

$$ff'' - (f')^2 = -g(g + 2zf), \quad (f'g - fg')^2 - 4f'f^2g = g^4 + 4zfg^3 + 4(z^2 - \alpha)f^2g^2 - 2\beta f^4.$$

References

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178 Painlevé equation P_5

$$u'' = \left(\frac{1}{2u} + \frac{1}{u-1}\right)(u')^2 - \frac{u'}{z} + \frac{(u-1)^2}{z^2}\left(\alpha u + \frac{\beta}{u}\right) + \gamma\frac{u}{z} + \delta\frac{u(u+1)}{u-1} \qquad (\mathrm{P}_5)$$

Representation by entire functions: $u = g/f$,

$$ff'' - (f')^2 = f(f'-g')^2 + 2\alpha g(g-f),$$
$$(f'g - fg')^2 = 2fg(f-g)(f'-g') + 2(\alpha g^2 - \beta f^2)(f-g)^2 + 2\gamma e^z f^2 g(f-g) - 2\delta e^{2z} f^2 g^2.$$

References

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179 Painlevé equation P_6

$$u'' = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-z}\right)(u')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{u-z}\right)u' + \frac{u(u-1)(u-z)}{z^2(z-1)^2}\left(\alpha + \beta\frac{z}{u^2} + \gamma\frac{z-1}{(u-1)^2} + \delta\frac{z(z-1)}{(u-z)^2}\right) \tag{P_6}$$

References

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180 Painlevé discrete equations

There exist a lot of nonautonomous difference equations which can be interpreted as discrete analogs of Painlevé equations. A comprehensive list can be found in [1]. Here we give several simplest examples.

Versions of dP_1, accordingly to [2]:

$$\frac{an+b}{u_{n+1}+u_n}+\frac{a(n-1)+b}{u_n+u_{n-1}}=c-u_n^2 \tag{1}$$

$$u_{n+1}+u_n+u_{n-1}=\frac{an+b}{u_n}+c \tag{2}$$

$$u_{n+1}+u_{n-1}=\frac{an+b}{u_n}+\frac{c}{u_n^2} \tag{3}$$

$$u_{n+1}+u_{n-1}=\frac{an+b}{u_n}+c \tag{4}$$

$$u_{n+1}u_{n-1}=\frac{e^{an+b}}{u_n}+\frac{c}{u_n^2} \tag{5}$$

Equation (2) was introduced in [3, 4].

Versions of dP_2:

$$u_{n+1}+u_{n-1}=\frac{(an+b)u_n+a}{u_n^2-1} \tag{6}$$

$$\frac{an+b}{u_nu_{n+1}+1}+\frac{a(n-1)+b}{u_{n-1}u_n+1}=\frac{1}{u_n}-u_n+an+b+c \tag{7}$$

$$u_{n+1}u_{n-1}=\alpha\frac{1+q^n/u_n}{1+q^nu_n} \tag{8}$$

Equation (6) was introduced in [5], in [6] the Miura-type transformation was found to the dP_{34}:

$$(u_{n+1}+u_n)(u_n+u_{n-1})=\frac{m^2-4u_n^2}{\lambda u_n+n\alpha+\beta+(-1)^n\gamma}.$$

Alternate dP_2 can be interpreted as nonlinear superposition principle for (P_3) [2].

dP_4 [7, 8]:

$$(u_{n+1}+u_n)(u_n+u_{n-1})=\frac{(u_n+\alpha+\beta)(u_n+\alpha-\beta)(u_n-\alpha+\beta)(u_n-\alpha-\beta)}{(u_n+\delta n+\varepsilon+\gamma)(u_n+\delta n+\varepsilon-\gamma)}$$

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181 Periodic closure

Periodic boundary conditions turn infinite differential-difference equations into finite-dimensional dynamical systems. Typically, this preserves the integrability, since the lattice zero curvature representation is transformed into a finite-dimensional Lax representation:

$$W_{n,x} = U_{n+1}W_n - W_nU_n, \quad U_{n+N} = U_n \qquad \Rightarrow \qquad \hat{W}_{n,x} = [U_n, \hat{W}_n], \quad \hat{W}_n = W_{n+N-1}\dots W_{n+1}W_n$$

and the trace $\operatorname{tr}\hat{W}_n$ becomes the generating function of the integrals of motion. Of course, the completeness of these integrals has to be proven for each example individually.

If the original lattice defines the Bäcklund auto-transformations of some equation then its periodic version defines some subclass of its solution. This possibility was proposed in the papers [1, 2], and then it was shown [3] that in the case of Schrödinger operator this construction leads exactly to the finite-gap solutions of KdV equation, see also [4].

The periodicity condition can be combined with any point transformation leaving the lattice invariant. In general, this spoils the integrability, and leads to the interesting examples which are exactly solvable in the quantum-mechanical sense and are related to Painlevé transcendents and their q-analogs [5, 3, 6].

➢ *Example.* Following [3], consider the periodic solutions of the dressing chain:

$$f'_n + f'_{n+1} = f_n^2 - f_{n+1}^2 + \alpha_{n+1}, \quad n \in \mathbb{Z}_N, \quad \varepsilon = -\alpha_1 - \dots - \alpha_N \neq 0. \tag{1}$$

Under this reduction, applying of the operators A_n^+ brings after N steps to the ψ-functions of the potential shifted by ε. This means, assuming the regularity of potential and the suitable asymptotics of ψ-functions, that the spectrum of such potential consists of N arithmetic progressions. This is illustrated by the figure below corresponding to $N = 3$. The eigenfunctions of the operator L_n are constructed with the help of mutually conjugated creation-annihilation operators of N-th order

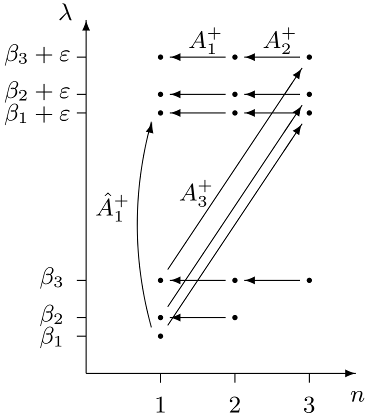
$$\hat{A}_n^+ = A_n^+ \dots A_{n+N-1}^+, \quad \hat{A}_n = A_{n+N-1}\dots A_n.$$

It is easy to prove that these operators satisfy the relations

$$\hat{A}_n^+\hat{A}_n = P(L_n), \quad \hat{A}_n\hat{A}_n^+ = P(L_n + \varepsilon), \quad P(\lambda) = (\lambda - \beta_n)\dots(\lambda - \beta_{n+N-1}),$$

$$[L_n, \hat{A}_n^+] = \varepsilon \hat{A}_n^+, \quad [L_n, \hat{A}_n] = -\varepsilon \hat{A}_n$$

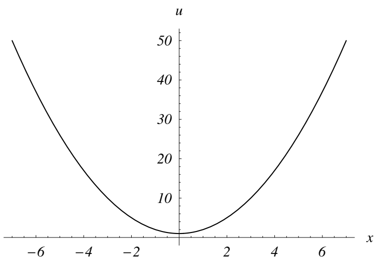
which generalize the algebra of harmonic oscillator which corresponds to $N = 1$. Of course, the question on the analytical properties of the system (1) solutions and the corresponding potentials and ψ-functions requires for an additional study.



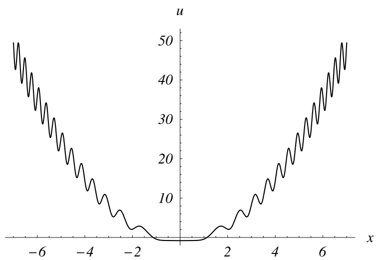
At $N = 3, 4$ the system (1) turns out to be equivalent to the Painlevé equations P_4 and P_5 respectively. It is likely that at $N \geq 5$ the system possesses the Painlevé property as well. However, for the spectral theory the qualitative information is of most importance about the regularity of potential and its asymptotics. The relation $2\sum f_n = -\varepsilon x$ suggests that

$$f_n = -\frac{\varepsilon x}{2N} + O(1), \quad u_n = \frac{\varepsilon^2 x^2}{4N^2} + O(x), \quad x \to \pm\infty.$$

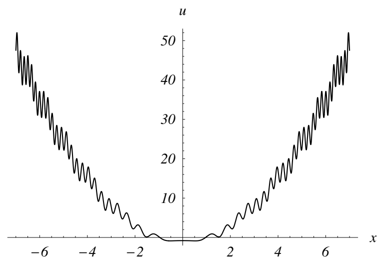
At odd N, the numerical experiments demonstrate that this asymptotics is true and the potential u_1 is regular on the whole axis for a rather large domain in the space of parameters and initial values of the system. In such cases, it is easy to prove that formula (37.3) provides the eigenfunctions of the operator L_1. The value of ε on the presented plots is chosen $\varepsilon = 2N$, so that the leading asymptotic term is x^2. The choice of initial values $f_n(0) = 0$ provides the even potential $u(-x) = u(x)$.



$N = 1$ (harmonic oscillator)

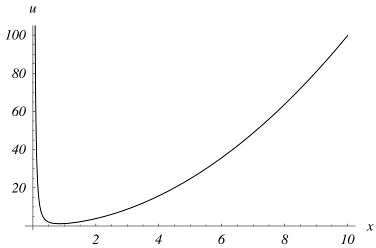


$N = 3$ (P_4)

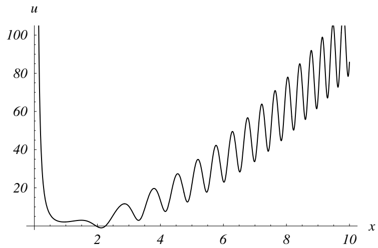


$N = 5$

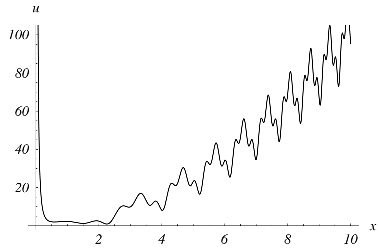
For even N, the potentials have a singularity at $x = 0$ so that the proper spectral problem is formulated on the halfline.



$N = 2$ (Kepler problem)



$N = 4$ (P_5)



$N = 6$

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182 Planar lattices

Author: V.E. Adler, Last. mod.: 1.12.2008

The following notion introduced in [1] ($M=2$) and [2] ($M>2$) is the most simple and fundamental model of integrable discrete geometry.

Definition 1. A mapping $f:\mathbb{Z}^M\to\mathbb{RP}^d$, $d>2$, is called M-dimensional ***planar lattice*** or ***discrete conjugate net***, if the image of any unit square in $\mathbb{Z}^M$ is a planar quadrangle.

In affine coordinates, a 2-dimensional planar lattice is uniquely defined (in the quadrant $\mathbb{Z}^2_+$) by equation of the form

$$(T_1-1)(T_2-1)f=c^{21}(T_1-1)f+c^{12}(T_2-1)f$$

with arbitrary scalar parameters $c^{12}(m,n)$, $c^{21}(m,n)$ and arbitrary initial values $f(n,0)$, $f(0,n)$ along the coordinate axes (clearly, the other settings of initial value problem are also possible).

At $M=3$, a planar lattice is uniquely defined by its values on the coordinate planes, that is by 2-dimensional planar lattices $f(m,n,0)$, $f(m,0,n)$ and $f(0,m,n)$. Indeed, let Π^{ij} denote 2-dimensional plane through the points f,f_i,f_j. Here and below in this section subscripts are used to denote shifts, that is $f_i=f(\dots,n_i+1,\dots)$. Consider three such planes Π^{ij}_k, $i\ne j\ne k\ne i$. By construction, these planes lie in 3-dimensional affine space Π^{123} through the points f,f_1,f_2,f_3 and therefore their intersection defines f_{123} uniquely.

From the computational point of view this means that we are able to satisfy simultaneously the linear equations

$$(T_i-1)(T_j-1)f=c^{ji}(T_i-1)f+c^{ij}(T_j-1)f,\quad i\ne j \tag{1}$$

for i,j taking values $1,2,3$. This yields the compatibility condition for the coefficients (no summation over repeated indices)

$$c^{ij}_k-c^{ij}=(c^{ik}_j-c^{ij}_k)c^{kj}+c^{ki}_jc^{ij},\quad i\ne j\ne k\ne i \tag{2}$$

which can be solved with respect to the shifted coefficients, so that some birational mapping

$$(c^{12},c^{21},c^{13},c^{31},c^{23},c^{32})\mapsto(c^{12}_3,c^{21}_3,c^{13}_2,c^{31}_2,c^{23}_1,c^{32}_1)$$

arises. This mapping is rather cumbersome (it is written in [3], in a slightly different notation), but, fortunately, there exists a change of variables which brings it to a very nice form. Namely, alternating (2) with respect to i, k yields the relation

$$(c^{ij}_k + 1)(c^{kj} + 1) = (c^{kj}_i + 1)(c^{ij} + 1)$$

which is solved by introducing the quantities h^i (attached to the directed edges of the lattice) accordingly to the formula

$$c^{ij} + 1 = h^j_i / h^j.$$

Now, introducing the vectors $v^i = (h^i)^{-1}(T_i - 1)f$ we bring the linear problem (1) to the form

$$(T_i - 1)v^j = \beta^{ji} v^i, \quad \beta^{ji} := \frac{(T_j - 1)h^i}{h^j_i}.$$

The new parameters β^{ij} are called ***discrete rotation coefficients***. Their evolution is given by equations (see also [4])

$$\beta^{kj}_i = \frac{\beta^{kj} + \beta^{ki}\beta^{ij}}{1 - \beta^{ij}\beta^{ji}}, \quad i \neq j \neq k \neq i. \tag{3}$$

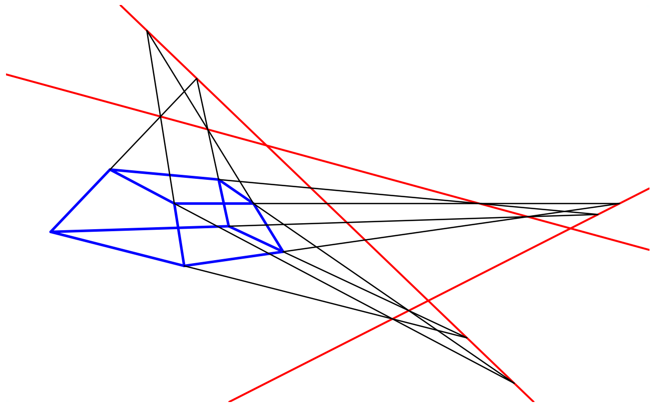
The important property of this evolution is that it can be correctly defined on the lattice of arbitrary dimension, so that the indices i, j, k may take arbitrary integer value and the commutativity property holds $\beta^{ij}_{kl} = \beta^{ij}_{lk}$. In other words, the map $\{\beta^{ij}\} \to \{\beta^{ij}_k\}$ is 4D-consistent. This can be easily verified directly, but more simple and profound proof follows from the underlying geometric picture.

Theorem 2 (Doliwa, Santini). *3-dimensional planar lattices are 4D-consistent.*

Proof. The initial data for a 4D cubic cell are vectors f, f_i, f_{ij}, $1 \le i < j \le 4$, such that $f_{ij} \in \Pi^{ij}$. This defines $f_{123} = \Pi^{12}_3 \cap \Pi^{13}_2 \cap \Pi^{23}_1$ and analogously for f_{124}, f_{134} and f_{234}. The value f_{1234} can be found as intersection $\Pi^{12}_{34} \cap \Pi^{13}_{24} \cap \Pi^{23}_{14}$, but obviously there are three more ways to do this. Therefore, we have to prove that six planes Π^{ij}_{kl} meet in one point. But $\Pi^{ij}_{kl} = \Pi^{ijk}_l \cap \Pi^{ijl}_k$, so that we have actually intersection of four 3D spaces in 4D space Π^{1234} which give us a unique point. ■

Notice, that this "general position" proof requires some modification in the case when all initial data lie in some 3D subspace (this may occur if the embedding dimension $d = 3$ or just as an accidental degeneration). This can be achieved by considering a 4D figure with the same 3D projection, like as in the proof of Desargues theorem in a plane. Moreover, similar trick allows to define quadrilateral lattice also in the case $d = 2$ when the Definition 1 makes no sense: it is sufficient to require that this lattice be a projection of some lattice in $\mathbb{RP}^3$. A method to draw such projection effectively is given by the following theorem.

Theorem 3 ([5]). *Consider a combinatorial cube on the plane. If, for some pair of the opposite faces, the (prolongations of) corresponding edges meet on a straight line, then the same is true for any other pair.*



Proof. Collinearity of one quadruple of the intersection points allows to construct a combinatorial cube in space, with planar faces, for which our figure is a projection. For such a figure, edges meet on the intersections of 3 pairs of the planes. ∎

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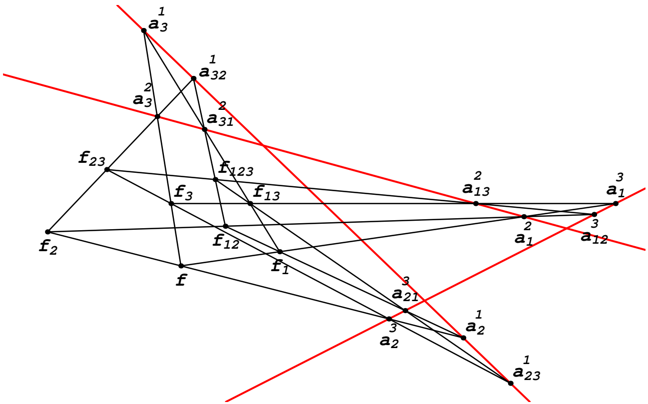

Remark 1. Collinearity of 4 intersection points is the condition, which allows to construct any vertex of the combinatorial cube by the other ones. This defines the mapping $(\mathbb{RP}^2)^7 \to \mathbb{RP}^2$. Let $f, f_1, \dots, f_{23}$ be given, then f_{123} is defined by

$$a^3_1 = ff_1 \cap f_3f_{13}, \quad a^3_2 = ff_2 \cap f_3f_{23}$$
$$a^3_{12} = f_2f_{12} \cap a^3_1a^3_2, \quad a^3_{21} = f_1f_{12} \cap a^3_1a^3_2$$
$$f_{123} = a^3_{12}f_{23} \cap a^3_{21}f_{13}.$$

The theorem means that the result is invariant with respect to the permutations of the subscripts.

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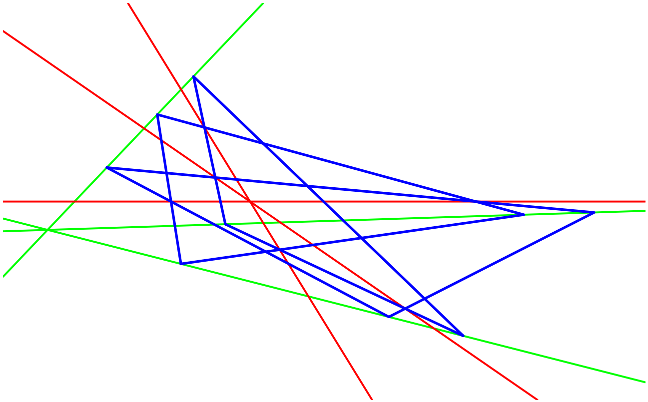

Remark 2. Notice that all lines and points in the above figure are on equal footing. Namely, 8 vertices of the cube + 12 intersection points and 12 sides + 3 lines of intersections form a *regular* configuration with the symbol $(20_3 15_4)$. This configuration is mentioned in [6], in connection with the following statement (equivalent to Theorem 3):

Let 3 triangles be perspective with the common center. Then 3 axes of perspective of 3 pairs of triangles meet in one point.

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183 Plebanski equations

first:	$u_{xy}u_{zt} - u_{xt}u_{zy} = 1$	(1)
second:	$u_{tx} + u_{zy} = u_{xy}^2 - u_{xx}u_{yy}$	(2)

Alias: heavenly equation

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184 Pohlmeyer–Lund–Regge system

$$s_{xy} + \langle s_x, s_y \rangle s = 0, \quad s \in \mathbb{R}^d, \quad |s| = 1 \tag{1}$$

Alias: $O(n)$ σ-model

➢ Due to the pseudo-constants $|s_x|_y = 0$, $|s_y|_x = 0$, the normalization

$$|s_x| = |s_y| = 1$$

can be achieved without loss of generality, by a change $x \to \tilde{x}(x)$, $y \to \tilde{y}(y)$.

➢ At $d = 3$, the substitution $\langle s_x, s_y \rangle = \cos u$ brings to the sine-Gordon equation $u_{xy} = -\sin u$.

➢ Similarly, at $d = 4$, the system appears [3]

$$u_{xy} - \sin u \cos u + \frac{\cos u}{\sin^3 u} v_x v_y = 0, \quad (v_y \cot^2 u)_x + (v_x \cot^2 u)_y = 0. \tag{2}$$

The reduction $v = 0$ brings to the sine-Gordon equation again. The system (2) can be cast to the rational form

$$u_{xy} = \frac{v u_x u_y}{uv + 1} + u(uv + 1), \quad v_{xy} = \frac{u v_x v_y}{uv + 1} + v(uv + 1) \tag{3}$$

via the point transformation.

➢ Bäcklund transformation for (3) [4]:

$$\begin{aligned} u_{n,x} &= (u_n v_n + 1) u_{n+1}, & u_{n,y} &= (u_n v_n + 1) u_{n-1} \\ -v_{n,x} &= (u_n v_n + 1) v_{n-1}, & -v_{n,y} &= (u_n v_n + 1) v_{n+1} \end{aligned}$$

These lattices belong to the hierarchy of Ablowitz–Ladik lattice. Their higher symmetries bring to NLS-type systems. This relation was studied also in [5].

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185 Pohlmeyer–Lund–Regge type systems

$$u_{xy} = f(u_x, u_y, u, v), \quad v_{xy} = g(v_x, v_y, v, u). \tag{1}$$

Main examples are:

$$u_{xy} = 2uvu_y - u, \quad v_{xy} = -2uvv_y - v \tag{2}$$

$$u_{xy} = h^{-1}u_xu_y + h(1-u_y), \quad v_{xy} = h^{-1}v_xv_y + h(1+v_y), \quad h = u+v \tag{3}$$

$$u_{xy} = h^{-1}u_x(vu_y - 1) + hu_y, \quad v_{xy} = h^{-1}v_x(uv_y + 1) - hv_y, \quad h = uv + \delta \tag{4}$$

$$u_{xy} = h^{-1}vu_xu_y - uh, \quad v_{xy} = h^{-1}uv_xv_y - vh, \quad h = uv - 1 \tag{5}$$

$$\begin{gathered} u_{xy} = h^{-1}(h_uu_xu_y + g(u_x+u_y) + g_vh - gh_v), \quad v_{xy} = h^{-1}(h_vv_xv_y - g(v_x+v_y) + g_uh - gh_u), \\ g = hh_{uv} - h_uh_v, \quad h(u,v) = h(v,u), \quad h_{uuu} = 0 \end{gathered} \tag{6}$$

➢ The Pohlmeyer–Lund–Regge system [1, 2] itself is point equivalent to the system (5).

➢ All above systems are Lagrangian, e.g. for the system (6) [4]:

$$L = \int\int h^{-1}(u_xv_y + h_uu_x - h_vv_y + g)dxdy.$$

➢ The complex reductions are possible, e.g. the system (2) turns into

$$u_{xy} = u - 2i|u|^2u_y$$

after the change $\partial_x \to i\partial_x$, $\partial_y \to i\partial_y$ and under the reduction $v = \bar{u}$.

At $h = \mathrm{const}(u-v)^2$ the system (6) turns into

$$u_{xy} = \frac{2u_xu_y}{u-v} - i(u_x+u_y), \quad v_{xy} = \frac{2v_xv_y}{v-u} + i(v_x+v_y). \tag{7}$$

➢ The following theorem establishes the correspondence between the PLR type systems and the lattices generated by Bäcklund transformations for NLS type systems.

Theorem 1. *The systems (2)–(6) are obtained by elimination of the shifts from the following consistent pairs of the lattices ($x_+ = x$, $x_- = y$):*

$$(2):\quad \begin{aligned} & u_{n,x} = u_{n+1} + u_n^2 v_n, \quad -v_{n,x} = v_{n-1} + v_n^2 u_n, \\ & u_{n,y} = \frac{u_{n-1}}{v_n u_{n-1} - 1}, \quad -v_{n,y} = \frac{v_{n+1}}{u_n v_{n+1} - 1} \end{aligned}$$

$$(3):\quad \begin{aligned} & u_{n,x} = (u_n + v_n)(u_{n+1} - u_n), \quad -v_{n,x} = (u_n + v_n)(v_{n-1} - v_n), \\ & u_{n,y} = \frac{u_n + v_n}{v_n + u_{n-1}}, \quad -v_{n,y} = \frac{u_n + v_n}{u_n + v_{n+1}} \end{aligned}$$

$$(4):\quad \begin{aligned} & u_{n,x} = (u_n v_n + \delta)(u_{n+1} + u_n), \quad -v_{n,x} = (u_n v_n + \delta)(v_{n-1} + v_n), \\ & u_{n,y} = \frac{u_n + u_{n-1}}{v_n u_{n-1} - \delta}, \quad -v_{n,y} = \frac{v_n + v_{n+1}}{u_n v_{n+1} - \delta} \end{aligned}$$

$$(5):\quad u_{n,x_\pm} = (u_n v_n - 1)u_{n\pm1}, \quad -v_{n,x_\pm} = (u_n v_n - 1)v_{n\mp1}$$

$$(6):\quad u_{n,x_\pm} = \frac{2h}{u_{n\pm1} - v_n} + h_{v_n}, \quad v_{n,x_\pm} = \frac{2h}{u_n - v_{n\mp1}} - h_{u_n}, \quad h = h(u_n, v_n).$$

These formulae can be interpreted as explicit Bäcklund transformations for the PLR type systems. For example, for the system (2) one obtains the pair of auto-transformations $(u, v) \to (u, v)_{\pm1}$

$$u_1 = u_x - u^2 v, \quad v_1 = \frac{v_y}{u v_y + 1},$$

$$u_{-1} = \frac{u_y}{v u_y - 1}, \quad v_{-1} = -v_x - v^2 u,$$

which are mutually inverse in virtue of the system.

➢ In conclusion, it should be noted that the full hierarchy of the higher symmetries for a PLR type system consists of two subhierarchies of NLS type, one contains the systems with x-derivatives, and another with the y-derivatives. For example, the simplest higher symmetries of (2) are

$$u_{t_1} = u_{xx} - 2(u^2 v_x + u^3 v^2), \quad -v_{t_1} = v_{xx} + 2(v^2 u_x - v^3 u^2)$$

$$u_{t_{-1}} = u_{yy} + 2u_y^2 v_y, \quad -v_{t_{-1}} = v_{yy} - 2v_y^2 u_y.$$

These are obtained by elimination of the shifts from the higher symmetries of the corresponding lattices.

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186 Point transformations

Let $u = (u^1, \dots, u^m)$ and $x = (x_1, \dots, x_n)$ be dependent and independent variables respectively. We will use the multi-index notation u_s, $s = (s_1, \dots, s_n)$ for the derivatives. A ***point transformation*** is defined by an arbitrary nondegenerate change

$$\tilde{x}_i = f_i(x, u), \quad \tilde{u}^j = g^j(x, u). \tag{1}$$

The prolongation of the transformation onto variables u_s is given by the formula

$$\tilde{u}^j_s = \tilde{D}^s(\tilde{u}^j) = \tilde{D}_1^{s_1} \cdots \tilde{D}_n^{s_n}(\tilde{u}^j), \tag{2}$$

where operators $\tilde{D}_i$ are related with the operators of the total derivatives

$$D_i = \partial_{x_i} + \sum_{j=1}^{m} \sum_{s} u^j_{s+1_i} \partial_{u^j_s}, \quad 1_i = (\delta_{1,i}, \dots, \delta_{n,i})$$

via the system of linear algebraic equations

$$D_i(\tilde{x}_1)\tilde{D}_1 + \cdots + D_i(\tilde{x}_n)\tilde{D}_n = D_i, \quad i = 1, \dots, n. \tag{3}$$

The determinant of this system does not vanish in virtue of the nondegeneracy of the transform (1). Moreover, equations (1), (2) define the invertible transformation on the set of the variables $J^r = \{x, u_s : |s| = s_1 + \cdots + s_n \le r\}$ for any r.

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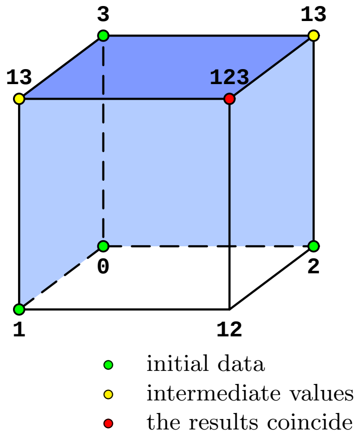
187 Quad-equations

Author: V.E. Adler, 21.07.2005; Last mod. 3.12.2008

Quad-equation is a discrete equation on the lattice $\mathbb{Z}^2$, which relates the values of a field variable corresponding to the vertices of any unit square. In a more general setup, the equations of quad-graphs are considered, that is on the planar graphs with quadrangle faces. Quad-equations appear as the nonlinear superposition principle for Bäcklund transformation. It is the commutativity of BTs what implies the $3D$-consistency property and motivates the acceptance of this property as an intrinsic definition of integrability for quad-equations.

1. $3D$-consistency



Denote the vertices of the cube as shown on the picture and consider the system of 6 quad-equations associated to the faces of the cube (assuming $u_{ij} := u_{ji}$):

$$Q_{ij}(u, u_i, u_j, u_{ij}) = 0, \quad Q_{ij}(u_k, u_{ik}, u_{jk}, u_{123}) = 0.$$

This system is called ***3D-consistent*** [1, 2], or ***consistent around the cube***, if the values u_{123} calculated in three possible ways coincide for any choice of initial data u, u_1, u_2, u_3.

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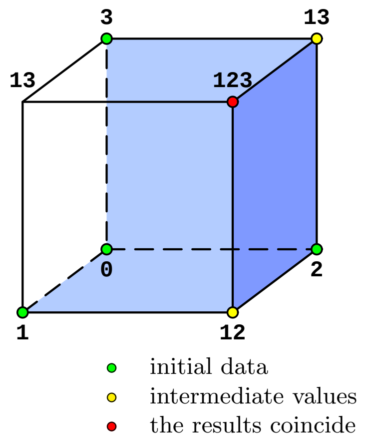

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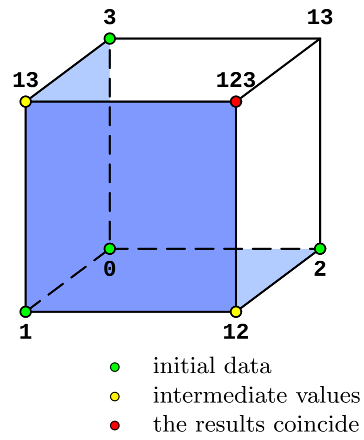

Example 1. Discrete KdV equation

$$(u - u_{ij})(u_i - u_j) = a_i - a_j, \quad (u_k - u_{123})(u_{ik} - u_{jk}) = a_i - a_j$$

(parameter a_i corresponds to 4 edges of the cube parallel to the edge $(0, i)$). One of the ways of computation yields

$$u_{12} = u - \frac{a_1 - a_2}{u_1 - u_2}, \quad u_{13} = u - \frac{a_1 - a_3}{u_1 - u_3},$$
$$u_{123} = u_1 - \frac{a_2 - a_3}{u_{12} - u_{13}} = \frac{a_1 u_1 (u_2 - u_3) + a_2 u_2 (u_3 - u_1) + a_3 u_3 (u_1 - u_2)}{a_1 (u_2 - u_3) + a_2 (u_3 - u_1) + a_3 (u_1 - u_2)}.$$

Since this expression is symmetric with respect to the subscripts, two another ways give the same result.

Example 2. Linear equation

$$u_{ij} - u_i - u_j + u = 0, \quad u_{123} - u_{ik} - u_{jk} + u_k = 0.$$

Independently on the order of computations $u_{123} = u_1 + u_2 + u_3 - 2u$.

2. List of quad-equations

The classification of $3D$-consistent equations has been obtained in [3] under the following assumptions:

➢ $Q_{ij}(u, u_i, u_j, u_{ij}) = Q(u, u_i, u_j, u_{ij}, a_i, a_j)$ where a_i are parameters assigned to the edges parallel to $(0, i)$;

➢ function Q is affine-linear polynomial in u: $Q = c_1 u u_1 u_2 u_{12} + \dots + c_{16}$ with coefficients depending on a_i;

➢ the equations admit the symmetry group of the square ($\varepsilon^2 = \sigma^2 = 1$):

$$Q(u, u_1, u_2, u_{12}, a_1 \alpha_2) = \varepsilon Q(u, u_2, u_1, u_{12}, a_2, a_1) = \sigma Q(u_1, u, u_{12}, u_2, a_1, a_2); \qquad (1)$$

➢ the ***tetrahedron condition***: u_{123} as the function on initial data does not depend on u (cf examples 1, 2).

Theorem 3. *Up to the simultaneous Möbius transformations of variables and point transformations of parameters 3D-consistent equations satisfying the above assumptions are exhausted by the following list:*

$$a_1(u-u_2)(u_1-u_{12})-a_2(u-u_1)(u_2-u_{12})=\delta^2a_1a_2(a_2-a_1) \qquad (\mathrm{Q}_1)$$

$$a_1(u-u_2)(u_1-u_{12})-a_2(u-u_1)(u_2-u_{12}) \qquad (\mathrm{Q}_2)$$
$$+a_1a_2(a_1-a_2)(u+u_1+u_2+u_{12})=a_1a_2(a_1-a_2)(a_1^2-a_1a_2+a_2^2)$$

$$(a_2^2-a_1^2)(uu_{12}+u_1u_2)+a_2(a_1^2-1)(uu_1+u_2u_{12})-a_1(a_2^2-1)(uu_2+u_1u_{12}) \qquad (\mathrm{Q}_3)$$
$$=\delta^2(a_1^2-a_2^2)(a_1^2-1)(a_2^2-1)/(4a_1a_2)$$

$$\operatorname{sn}a_1\operatorname{sn}a_2\operatorname{sn}(a_1-a_2)(k^2uu_1u_2u_{12}+1)+\operatorname{sn}a_1(uu_1+u_2u_{12}) \qquad (\mathrm{Q}_4)$$
$$-\operatorname{sn}a_2(uu_2+u_1u_{12})-\operatorname{sn}(a_1-a_2)(uu_{12}+u_1u_2)=0, \qquad \operatorname{sn}a\equiv\operatorname{sn}(a;k)$$

$$(u-u_{12})(u_1-u_2)=a_1-a_2 \qquad (\mathrm{H}_1)$$

$$(u-u_{12})(u_1-u_2)+(a_2-a_1)(u+u_1+u_2+u_{12})=a_1^2-a_2^2 \qquad (\mathrm{H}_2)$$

$$a_1(uu_1+u_2u_{12})-a_2(uu_2+u_1u_{12})=\delta(a_2^2-a_1^2) \qquad (\mathrm{H}_3)$$

$$a_1(u+u_2)(u_1+u_{12})-a_2(u+u_1)(u_2+u_{12})=\delta^2a_1a_2(a_1-a_2) \qquad (\mathrm{A}_1)$$

$$(a_2^2-a_1^2)(uu_1u_2u_{12}+1)=a_1(a_2^2-1)(uu_1+u_2u_{12})-a_2(a_1^2-1)(uu_2+u_1u_{12}) \qquad (\mathrm{A}_2)$$

The proof is based on the relations between affine-linear, biquadratic and 4-th degree polynomials. Under the imposed assumptions the relation holds

$$Q_{u_2}Q_{u_{12}}-QQ_{u_2u_{12}}=k(a_1,a_2)h(u,u_1,a_1)$$

where

$$k(a_2,a_1)=-k(a_1,a_2), \quad h(u_1,u,a_1)=h(u,u_1,a_1),$$

and moreover, the biquadratic h is such that the 4-th order polynomial

$$h_{u_1}^2-2hh_{u_1u_1}=r(u)$$

does not depend on the parameters of equation. After this, the classification is reduced to the problem of reconstruction of h and Q starting from the polynomial r which can be brought to some canonical form by Möbius transformations.

Remark 4. ➢ Eq (A_1) is reduced to (Q_1) by the change $u_i \to -u_i$; (A_2) is reduced to (Q_3) by the change $u_i \to 1/u_i$.

➢ Eqs (Q_1)–(Q_3) and (H_1), (H_2) can be obtained from (Q_4), (H_3) by degenerations and as limiting cases.

➢ Eq (Q_4) defines the nonlinear superposition principle for the Krichever–Novikov equation and is, in a sense, the fundamental discrete equation [4].

➢ The given form of (Q_4) is found by Hietarinta [SIDE-2004 talk]. In [3] this equation was presented in much more cumbersome form related to the Weierstrass form of elliptic curve $A^2 = r(a) = 4a^3 - g_2a - g_3$.

➢ The problem of classification without additional assumptions (affine-linearity, prescribed dependence on parameters, symmetry, tetrahedron property) remains open. In particular, several examples without tetrahedron property were found in [5]. It can be proved that the biquadratics h corresponding to such equations are reducible.

➢ Several equations are known with polynomial Q quadratic in each variable, but all these examples can be reduced to affine-linear ones by Miura type transformations.

3. Zero curvature representation

An affine-linear equation $Q = 0$ may be interpreted as Möbius transformation between any pair of variables, with coefficients depending on the rest pair. Let

$$u_{13} = M(u_1, u, a_1, a_3; u_3) = \frac{Au_3 + B}{Cu_3 + D}$$

then

$$u_{23} = M(u_2, u, a_2, a_3; u_3), \quad u_{123} = M(u_{12}, u_2, a_1, a_3; u_{23}) = M(u_{12}, u_1, a_2, a_3; u_{13}).$$

Since the composition of Möbius transformations corresponds to the product of the matrices, hence denoting $a_3 \to \lambda$ and introducing the normalization factor yields the zero curvature representation

$$L(u_{12}, u_1, a_2, \lambda)L(u_1, u, a_1, \lambda) = L(u_{12}, u_2, a_1, \lambda)L(u_2, u, a_2, \lambda)$$

with the matrix

$$L(u_1, u, a_1, \lambda) = (AD - BC)^{-1/2} \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

For example, in the case of the discrete KdV equation (H1) one obtains

$$L(u_1,u,a_1,\lambda)=\begin{pmatrix} u & -uu_1+a_1-\lambda \\ 1 & -u_1 \end{pmatrix}.$$

4. Three-leg form and discrete Toda lattices

Let the quad-equation $Q(u,u_1,u_2,u_{12},a_1,a_2)=0$ possesses the square symmetry (1). We will say that it admits ***three-leg form*** if it is equivalent to the equation of the form

$$\phi(u,u_{12},a_1,a_2)=\psi(u,u_1,a_1)-\psi(u,u_2,a_2).$$

Often it is convenient to use alternatively the multiplicative three-leg form

$$F(x,x_{12},\alpha_1-\alpha_2)=F(x,x_1,\alpha_1)/F(x,x_2,\alpha_2).$$

Any three-leg equation corresponds to a discrete Toda lattice on a planar graph

$$\sum_n \phi(u,u_{n,n+1},a_n,a_{n+1})=0$$

where the sum is taken over the star of the vertex u.

Three-leg form exists for all equations from the above list, see the table. The general formula can be proved

$$\psi(u,u_1,a_1)=\int\frac{du_1}{h(u,u_1,a_1)}+C(u,a_1).$$

For the equations (Q_n), a point change of parameters $a=a(\alpha)$ exists such that $\phi(u,u_{12},a_1,a_2)=\psi(u,u_{12},a(\alpha_1-\alpha_2))$. Moreover, it is often convenient to make changes of the variables $u=u(x)$ as well.

	$F(x,y,\alpha)$	$u=u(x)$	$a=a(\alpha)$
$(Q_1)_{\delta=0}$	$\exp(\alpha/(x-y))$	x	α
$(Q_1)_{\delta=1}$	$\dfrac{x-y+\alpha}{x-y-\alpha}$	x	α
(Q_2)	$\dfrac{(x+y+\alpha)(x-y+\alpha)}{(x+y-\alpha)(x-y-\alpha)}$	x^2	α
$(Q_3)_{\delta=0}$	$\dfrac{\sinh(x-y+\alpha)}{\sinh(x-y-\alpha)}$	$\exp 2x$	$\exp 2\alpha$
$(Q_3)_{\delta=1}$	$\dfrac{\sinh(x+y+\alpha)\sinh(x-y+\alpha)}{\sinh(x+y-\alpha)\sinh(x-y-\alpha)}$	$\cosh 2x$	$\exp 2\alpha$
(Q_4)	$\dfrac{\operatorname{sn}(x+\alpha)-\operatorname{sn} y}{\operatorname{sn}(x-\alpha)-\operatorname{sn} y}\cdot\dfrac{\Theta_4(x+\alpha)}{\Theta_4(x-\alpha)}$	$\operatorname{sn} x$	α

$$(H_1):\quad \frac{a_1-a_2}{u-u_{12}}=u_1-u_2,\qquad (H_2):\quad \frac{u-u_{12}+a_1-a_2}{u-u_{12}-a_1+a_2}=\frac{u+u_1+a_1}{u+u_2+a_2}$$

$$(H_3):\quad \frac{a_2u-a_1u_{12}}{a_1u-a_2u_{12}}=\frac{uu_1+\delta a_1}{uu_2+\delta a_2}$$

Remark 5. For the eq (Q_4) with the polynomial r in Weierstrass form, the leg is

$$F=\frac{\sigma(x+y+\alpha)\sigma(x-y+\alpha)}{\sigma(x+y-\alpha)\sigma(x-y-\alpha)}.$$

5. Multifield quad-equations

Classification of multifield analogs of quad-equations is hardly possible. One of the reasons is that these equations are not polynomial, in contrast to the scalar case. Probably, the simplest example is the vector analog of the discrete KdV eq:

$$u - u_{12} = \frac{a_1 - a_2}{|u_1 - u_2|^2}(u_1 - u_2).$$

This equation admits an interesting reduction $a_i = -|u_i - u|^2$ [6]. Some other examples can be found in [7, 8].

The nonabelian analogs for the Krichever–Novikov equation (121.1) are known only for few special cases:

➢ $r = 0$ Schwarz-KdV). The equation, its BT and NSP are

$$u_{t_3} = u_{xxx} - \frac{3}{2}u_{xx}u_x^{-1}u_{xx}, \qquad u_{i,x} = a_i(u - u_i)u_x^{-1}(u - u_i)$$
$$a_1(u - u_2)(u_2 - u_{12})^{-1} = a_2(u - u_1)(u_1 - u_{12})^{-1}$$

➢ $r = 4$

$$u_{t_3} = u_{xxx} - \frac{3}{2}u_{xx}u_x^{-1}u_{xx} + 6u_x^{-1} + 3[u_x^{-1}, u_{xx}], \qquad u_{i,x} = \frac{1}{a_i}(u - u_i + a_i)u_x^{-1}(u - u_i - a_i)$$
$$a_1(u_1 - u_{12} + a_2)(u - u_1 - a_1)^{-1} = a_2(u_2 - u_{12} + a_1)(u - u_2 - a_2)^{-1}$$

➢ $r = u^2$

$$u_{t_3} = u_{xxx} - \frac{3}{2}(u_{xx}u_x^{-1}u_{xx} + u_{xx}u_x^{-1}u - uu_x^{-1}u_{xx} - uu_x^{-1}u)$$
$$u_{i,x} = \frac{1}{1 - a_i^2}(u - a_iu_i)u_x^{-1}(a_iu - u_i)$$
$$(1 - a_1^2)(u_1 - a_2u_{12})(a_1u - u_1)^{-1} = (1 - a_2^2)(u_2 - a_1u_{12})(a_2u - u_2)^{-1}$$

These equations possess also 3-legs forms which lead to nonabelian discrete Toda type lattices.

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188 Quispel–Roberts–Thompson mapping

$$f_3(u_n)u_{n+1}u_{n-1} - f_2(u_n)(u_{n+1} + u_{n-1}) + f_1(u_n) = 0, \quad \deg f_i \le 4 \tag{1}$$

Let $A(u, v)$, $B(u, v)$ be polynomials of degree 2 with respect to each variable:

$$A = a_1u^2v^2 + \cdots + a_9, \quad B = b_1u^2v^2 + \cdots + b_9.$$

Consider the mapping $(u_n, v_n) \to (u_{n+1}, v_{n+1})$ defined by equations

$$\frac{A(u_n, v_n)}{B(u_n, v_n)} = \frac{A(u_n, v_{n+1})}{B(u_n, v_{n+1})} = \frac{A(u_{n+1}, v_{n+1})}{B(u_{n+1}, v_{n+1})}.$$

We have to solve a quadratic equation at each step, however one of the roots is already known and therefore a birational mapping appears. The quantity $I = A/B$ is its invariant by construction and this provides the integrability. The mapping (1) appears when the polynomials A, B are symmetric.

➢ A particular case is the Euler–Chasles corrspondence (A is a symmetric polynomial and $B = 1$) [3, 4]. The simplest and, historically, the first example is McMillan map [5]

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189 Recursion operator

A differential or pseudo-differential operator R is called ***recursion operator*** [1] for an equation $E=0$ if it maps evolutionary symmetries of this equation into the evolutionary symmetries: $G\in\mathrm{Sym}(E) \Rightarrow R(G)\in\mathrm{Sym}(E)$.

➢ In particular, if equation E is evolutionary itself, that is of the form $u_t=F$, then equations $u_{t_k}=R^k(F)$, $k=1,2,3,\dots$ should be its symmetries. In this case the recursion operator satisfies the equation

$$R_t=[F_*,R],$$

which coincides with the equation for the formal symmetry. The difference between these two notions is that the formal symmetry is in general an infinite series and its action on $\mathrm{Sym}(E)$ is not defined. In contrast, recursion operator must act on this set by definition. As a rule, the recursion operator is represented as a ratio of differential operators $R=JK^{-1}$, and it is necessary to prove that the result of its application to the local symmetries is local again. See an example of such a proof for KdV equation.

The structure underlying the existence of recursion operator is given by Lenard–Magri scheme for the operators J,K which constitute a bi-Hamiltonian pair).

➢ See also: many examples of recursion operators are given in the corresponding articles, see e.g. Burgers equation, NLS system and so on.

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190 Reduction

Reduction is the lowering of the number of dependent or independent variables by means of additional constraints. The most general method of finding the reductions is based on invariance of the system under scrutiny with respect to some subgroup of continuous or discrete symmetries.

191 Reyman system, twodimensional

$$u_t = (u_x + u^2 - 2w_x)_x, \quad v_t = (-v_x + 2uv)_x, \quad w_y = v$$

➢ Master-symmetry:

$$u_\tau = (x(u_x + u^2 - 2w_x) - w)_x, \quad v_\tau = (x(-v_x + 2uv) - v)_x$$

➢ Third order flow $D_{t_3} = \frac{1}{2}[D_\tau, D_t]$:

$$u_{t_3} = (u_{xx} + 3uu_x + u^3 - 3uw_x - 3q_x)_x, \quad v_{t_3} = (v_{xx} - 3uv_x + 3u^2v - 3vw_x)_x, \quad q_y = uv.$$

➢ Auxiliary linear problems:

$$\psi_{xy} = u\psi_y + v\psi, \quad \psi_T = A(x)\psi_{xx} - (2Aw_x + A_xw)\psi$$

where $A = 1$ and $A = x$ correspond to $T = t$ and $T = \tau$ respectively. This is gauge equivalent to the linear problem

$$\psi_y = U\phi, \quad \phi_x = -V\psi, \quad u = \frac{U_x}{U}, \quad v = -UV$$

for Davey–Stewartson system.

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192 Relativistic Toda type lattices

$$z_{n,x} = r(z_n)(z_{n+1}f_n(y_n) - z_{n-1}f_{n-1}(y_{n-1}) + g_n(y_n) - g_{n-1}(y_{n-1})), \quad z_n := q_{n,x}, \; y_n := q_{n+1} - q_n \tag{1}$$

This is the Euler equation for the Lagrangian of the form

$$L = c(q_{n,x}) - q_{n,x}a_n(q_{n+1} - q_n) - b_n(q_{n+1} - q_n), \qquad r = 1/c'', \quad f_n = a'_n, \quad g_n = b'_n.$$

The lattice (1) is integrable if and only if

$$r(z) = r_2z^2 + r_1z + r_0, \quad f'_n = s_1f_n - r_1f_n^2 + 2r_2f_ng_n, \quad g'_n = s_0 + s_1g_n + r_2g_n^2 - r_0f_n^2.$$

In particular, the simplest higher symmetry is of the form

$$q_{n,t} = r(z_n)(z_{n+1}f_n(y_n) + z_{n-1}f_{n-1}(y_{n-1}) + g_n(y_n) + g_{n-1}(y_{n-1})) + s_1z_n^2. \tag{2}$$

The f, g-system is reduced to the equation

$$(f'_n)^2 = (r_1^2 - 4r_2r_0)f_n^4 + (4\alpha_nr_2 - 2s_1r_1)f_n^3 + (s_1^2 - 4r_2s_0)f_n^2$$

which is solved in elementary functions due to the first integral

$$(r_2g_n^2 + s_1g_n + s_0)/f_n - r_1g_n + r_0f_n =: \alpha_n.$$

If f, g do not depend on n, then the integrable lattice (1) can be brought to one of the following forms by means of the transformations $q_n \to c_1q_n + c_2t + c_3n$, $x \to c_4x$:

$$z_{n,x} = z_{n+1}e^{y_{n+1}} - z_{n-1}e^{y_n} - e^{2y_{n+1}} + e^{2y_n} \tag{3a}$$

$$z_{n,x} = z_n\Big(\frac{z_{n+1}}{y_{n+1}} - \frac{z_{n-1}}{y_n} + y_{n+1} - y_n\Big) \tag{3b}$$

$$z_{n,x} = z_n\Big(\frac{z_{n+1}}{1 + \mu e^{-y_{n+1}}} - \frac{z_{n-1}}{1 + \mu e^{-y_n}} + \nu(e^{y_{n+1}} - e^{y_n})\Big) \tag{3c}$$

$$z_{n,x} = z_n(z_n + 1)\Big(\frac{z_{n+1}}{y_{n+1}} - \frac{z_{n-1}}{y_n}\Big) \tag{3d}$$

$$z_{n,x} = z_n(z_n - \mu)\Big(\frac{z_{n+1}}{\mu + e^{y_{n+1}}} - \frac{z_{n-1}}{\mu + e^{y_n}}\Big) \tag{3e}$$

$$z_{n,x} = (z_n^2 + \mu)\Big(\frac{z_{n+1} - y_{n+1}}{\mu + y_{n+1}^2} - \frac{z_{n-1} - y_n}{\mu + y_n^2}\Big) \tag{3f}$$

$$z_{n,x} = \frac{1}{2}(z_n^2 + 1 - \mu^2)\Big(\frac{z_{n+1} - \sinh y_{n+1}}{\mu + \cosh y_{n+1}} - \frac{z_{n-1} - \sinh y_n}{\mu + \cosh y_n}\Big) \tag{3g}$$

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193 Rosenau–Hyman equation

$$u_t = uu_{xxx} + 3u_x u_{xx} + uu_x$$

This equation admits an exact travelling wave solution with compact support, known as ***compacton***:

$$u(x,t) = \begin{cases} -8a\cos^2 \frac{x-3at}{4}, & |x-3at| \le 2\pi, \\ 0, & |x-3at| \ge 2\pi. \end{cases}$$

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194 Rosochatius system

$$\ddot{q}_k = \dot{p}_k = -\omega_k q_k + \frac{\mu_k^2}{q_k^3} - q_k \sum_{j=1}^{N} \Big(\dot{q}_j^2 - \omega_j q_j^2 + \frac{\mu_j^2}{q_j^2} \Big), \quad \langle q, q \rangle = 1, \quad q = (q_1, \dots, q_N)^\intercal$$

➢ The Poisson structure is defined as the Dirac reduction of the canonical bracket to the level set $\langle q, q \rangle = 1$, $\langle q, p \rangle = 0$:

$$\{q_k, q_j\} = 0, \quad \{p_k, q_j\} = \delta_{kj} - q_k q_j, \quad \{p_k, p_j\} = q_k p_j - q_j p_k, \qquad H = \frac{1}{2} \sum_{k=1}^{N} \left(p_k^2 + \omega_k q_k^2 + \frac{\mu_k^2}{q_k^2} \right).$$

The $N-1$ independent first integrals in involution (assuming $\omega_k \neq \omega_j$, $\forall\ k, j$) are:

$$F_k = q_k^2 + \sum_{j \neq k} \frac{1}{\omega_k - \omega_j} \left((p_k q_j - p_j q_k)^2 + \frac{\mu_k^2 q_j^2}{q_k^2} + \frac{\mu_j^2 q_k^2}{q_j^2} \right), \quad \sum_{k=1}^{N} F_k = \langle q, q \rangle = 1, \quad \sum_{k=1}^{N} \omega_k F_k = H.$$

➢ Lax pair $\dot{L} = [M, L]$:

$$L = -\operatorname{diag}(\omega_1, \dots, \omega_N) + \lambda \left(p q^\intercal - q p^\intercal + i \frac{\mu}{q} q^\intercal + i q \Big(\frac{\mu}{q} \Big)^\intercal \right) + \lambda^2 q q^\intercal, \quad M = \lambda q q^\intercal + i \operatorname{diag}\Big(\frac{\mu_1}{q_1^2}, \dots, \frac{\mu_N}{q_N^2} \Big)$$

where $p, q, \dfrac{\mu}{q}$ are column vectors with the k-th entry p_k, q_k, $\dfrac{\mu_k}{q_k}$ respectively.

➢ See also Wojciechowski system

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195 Ruijsenaars–Schneider system

$$\ddot{u}_k = \sum_{j\neq k} \frac{\dot{u}_k \dot{u}_j f'(u_k - u_j)}{c - f(u_k - u_j)}, \quad j,k = 1,\dots,n, \qquad f(x) = \begin{cases} x^{-2} & \text{rational case} \\ \sinh^{-2} x & \text{hyperbolic case} \\ \wp(x) & \text{elliptic case} \end{cases}$$

➢ Lax pair was found in [2] (notice the functional equations

$$\alpha(x)\alpha'(y) - \alpha(y)\alpha'(x) = (\alpha(x+y) - \alpha(x)\alpha(y))(\eta(x) - \eta(y)),$$
$$\alpha(x+y) = \alpha(x)\alpha(y) + \phi(x)\phi(y)\psi(x+y)$$

solved in this work).

➢ See also Calogero–Moser model

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196 Sawada–Kotera equation

$$u_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1 \tag{1}$$

➢ Generic 3rd and 5th order operators L, A satisfying the Lax equation $L_t = [A, L]$ can be brought to the form

$$L = D_x^3 + uD_x + v, \quad -A = 9D_x^5 + 15uD_x^3 + 15(u_1 + v)D_x^2 + 5(2u_2 + u^2 + 3v_1)D_x + 10(v_2 + uv),$$

with u, v governed by the system

$$u_t = u_5 + 5(uu_2 + 3v(u_1 - v) + \frac{1}{3}u^3)_x,$$
$$v_t = v_5 + 5(uv_2 + 2u_2v + 2u_1v_1 - 3vv_1 + u^2v)_x.$$

This system admits the reductions:
$v = 0$, $v = u_x$ both corresponding to equation (1);
$2v = u_x$ corresponding to Kaup–Kupershmidt equation.

➢ Bäcklund transformation ($u = 6w_x$):

$$(\bar{w} - w)_{xx} + 3(\bar{w} - w)(\bar{w} + w)_x + (\bar{w} - w)^3 = \beta$$

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197 Sawada–Kotera equation, twodimensional

$$u_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1 - 5u_{2,y} - 5uu_y - 5u_1w - 5w_y, \quad u_y = w_x \tag{1}$$

➢ Introduced in [1].

➢ Consider auxiliary linear problems $\psi_y = L\psi$, $\psi_t = A\psi$. Generic 3rd and 5th order operators L, A can be brought to the form

$$L = D_x^3 + uD_x + v, \quad -A = 9D_x^5 + 15uD_x^3 + 15(u_1 + v)D_x^2 + 5(2u_2 + u^2 + 3v_1 + w)D_x + 5(2v_2 + 2uv + s),$$

with u, v governed by the system

$$u_t = u_5 + 5(uu_2 + 3v(u_1 - v) + \frac{1}{3}u^3)_x - 5(u_{2,y} + uu_y + u_1w + w_y), \quad u_y = w_x,$$
$$v_t = v_5 + 5(uv_2 + 2u_1v_1 + 2u_2v - 3vv_1 + u^2v)_x - 5(v_{2,y} + 2vu_y + uv_y + v_1w + s_y), \quad v_y = s_x.$$

This system admits the reductions:

$v = 0$, $s = 0$ and $v = u_x$, $s = u_x$ both corresponding to 2D-SK equation (1);

$2v = u_x$, $2s = u_y$ corresponding to 2D Kaup–Kupershmidt equation.

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198 Selfsimilar solutions

Selfsimilar solution of a PDE or DΔE is a special solution characterized by its invariance with respect to some subgroup of the Lie group of classical symmetries of the equation [1, 2, 3].

In the most common situations selfsimilar solutions are invariant with respect to some shift or dilation transformation. In many physical models such solutions define the asymptotic behaviour of the general solutions.

The construction of the selfsimilar solutions amounts to solving of the equation for the invariants of the subgroup. This reduces the dimensionality of the problem, for example, the selfsimilar solutions of an equation with two independent variables are defined by some ODEs. If the original equation was integrable then, accordingly to the Ablowitz–Ramani–Segur conjecture, these ODEs possess the Painlevé property.

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199 Shabat equation

$$q^2v'(qx) + v'(x) = (qv(qx) - v(x))^2 - 1, \quad v(0) = \alpha \tag{1}$$

➢ This ODE with proportional delay arises as the self-similar reduction of the dressing chain. Its solution is unique in the class of meromorphic function in $\mathbb{C}$. The spectrum of the corresponding Schrödinger operator with the potential $u = 2v'$ consists of the infinite geometric progression $-q^{2n}$, $n = 0, 1, \dots$ [2, 3].

➢ The analytic properties of the solution were studied in [4, 5, 6]. Rational solutions, corresponding to the special values of α were constructed in [7]. Some generalizations corresponding to selfsimilar closure of the dressing chain after several steps were discussed in [8].

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200 Sine-Gordon equation

$$u_{xy} = \sin u$$

➢ Introduced in [1].

➢ Bäcklund transformation [2]:

$$\hat{u}_x + u_x = 2a \sin \frac{\hat{u} - u}{2}, \quad \hat{u}_y - u_y = \frac{2}{a} \sin \frac{\hat{u} + u}{2}$$

➢ Zero curvature representation [4]:

$$U = \frac{1}{2}\begin{pmatrix} \lambda & -u_x \\ u_x & -\lambda \end{pmatrix}, \quad V = \frac{1}{2\lambda}\begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix}, \quad W = \begin{pmatrix} \lambda + a\cos\frac{\hat{u}-u}{2} & -a\sin\frac{\hat{u}-u}{2} \\ a\sin\frac{\hat{u}-u}{2} & -\lambda + a\sin\frac{\hat{u}-u}{2} \end{pmatrix}$$

➢ ***Kinks and breathers of sine-Gordon equation***

The equation and BT can be brought to the rational form by the change $z = \exp(iu/2)$:

$$zz_{xy} - z_x z_y = \frac{1}{4}(z^4 - 1), \quad (z\hat{z})_x = \frac{a}{2}(\hat{z}^2 - z^2), \quad z\hat{z}_y - z_y\hat{z} = \frac{1}{2a}(z^2\hat{z}^2 - 1), \quad a_1(zz_2 - z_1 z_{12}) = a_2(zz_1 - z_2 z_{12}).$$

The latter equation defines the nonlinear superposition and coincides with equation ($H_3|_{\delta=0}$). However, if we are interested in the real solutions, these formulae are better suited for the hyperbolic version of equation, $u_{xy} = \sinh u$, $z = \exp(u/2)$. In the trigonometric case, the reality is restored by the additional Möbius change $(z-1)/(z+1) = iv \Rightarrow v = \tan(u/4)$. This yields

$$v_{xy} = \frac{v}{1+v^2}(2v_x v_y + 1 - v^2), \quad (1+v^2)\hat{v}_x + (1+\hat{v}^2)v_x = a(\hat{v} - v)(1 + v\hat{v}), \quad (1+v^2)\hat{v}_y - (1+\hat{v}^2)v_y = \frac{1}{a}(\hat{v}+v)(1 - v\hat{v})$$

and the result of two BT is given by the formula

$$v_{12} = \frac{(a_1 - a_2)v(1 + v_1 v_2) - (a_1 + a_2)(v_1 - v_2)}{(a_1 - a_2)(1 + v_1 v_2) + (a_1 + a_2)v(v_1 - v_2)}. \tag{1}$$

Applying the BT to the seed solution $u=0$ we obtain immediately the kink solution

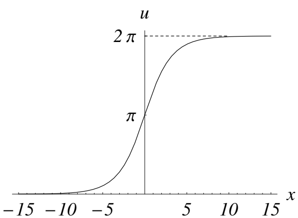
$$v=\tan(u/4)=c\exp(ax+y/a),$$

and the formula (1) allows to construct the multi-kinks. The general formula for 2-kink solution is

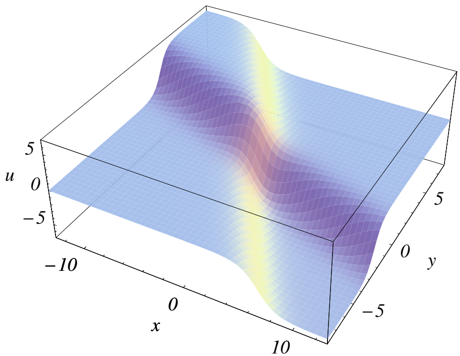
$$\tan\frac{u}{4}=\frac{(a_1+a_2)(c_2\exp(a_2x+y/a_2)-c_1\exp(a_1x+y/a_1))}{(a_1-a_2)(1+c_1c_2\exp((a_1+a_2)x+y(1/a_1+1/a_2)))}.$$

The solution v_{12} remains real if the intermediate solutions v_1,v_2 are complex conjugate. In particular, assuming in the above solution $a_1=\alpha+i\beta$, $a_2=\alpha-i\beta$ and $c_1=\bar{c}_2$ we come to the breather solution

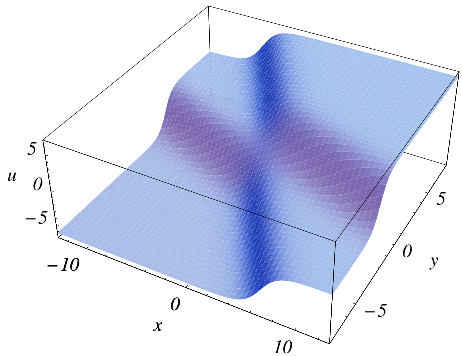
$$\tan\frac{u}{4}=\frac{\alpha\sin(\beta(x-y/\gamma)+\phi_1)}{\beta\cosh(\alpha(x+y/\gamma)+\phi_2)},\quad \gamma=\alpha^2+\beta^2.$$



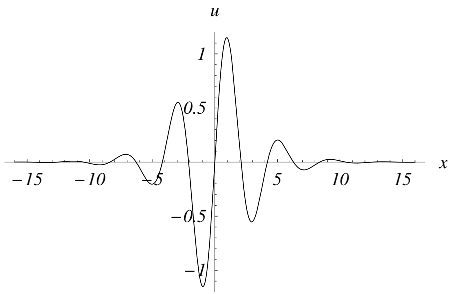
one-kink profile
$a=0.5,\ c=1,\ y=0$



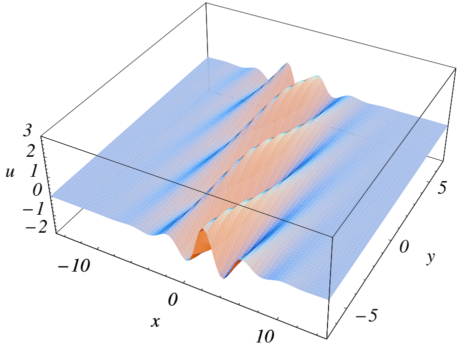
kink-antikink solution
$a_1=0.5,\ a_2=1,\ c_1=c_2=0$



two-kink solution
$a_1=0.5,\ a_2=-1,\ c_1=c_2=0$



breather profile $(t = 0)$



$\alpha = 0.5,\ \beta = 1.5,\ \phi_1 = \phi_2 = 0$

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201 Sine-Gordon equation, double

$$u_{xt} = \sin u + a \sin \frac{u}{2}$$

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202 Sine-Gordon equation, multidimensional

$$u_{x_1x_1} + \cdots + u_{x_nx_n} = \sin u$$

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203 Sklyanin lattice

The Sklyanin lattice [1] is defined by the Poisson brackets

$$\{s_n^a, s_n^0\} = (J_b - J_c)s_n^b s_n^c, \quad \{s_n^a, s_n^b\} = -s_n^0 s_n^c \tag{1}$$

(only nonzero values are given; $n \in \mathbb{Z}$, subscripts a, b, c form a cyclic permutation of $1, 2, 3$) and the Hamiltonian

$$H = \sum_n \log\Big(s_n^0 s_{n+1}^0 + \sum_{a=1}^3 \Big(\frac{c_1}{c_0} - J_a\Big) s_n^a s_{n+1}^a\Big), \quad c_0 = \sum_{a=1}^3 (s_n^a)^2, \quad c_1 = (s_n^0)^2 + \sum_{a=1}^3 J_a (s_n^a)^2.$$

The quantities c_0, c_1 are Casimir functions of this Poisson structure.

It was shown in [2] that Sklyanin lattice is equivalent to the sum of commuting flows

$$u_{n,x_\pm} = \frac{2h_n}{u_{n\pm1} - v_n} + h_{n,v_n}, \quad v_{n,x_\pm} = \frac{2h_n}{u_n - v_{n\mp1}} - h_{n,u_n}, \quad h_n = h(u_n, v_n) \tag{2}$$

where $h(u, v)$ is a symmetric biquadratic polynomial: $h(u, v) = h(v, u)$, $h_{uuu} = 0$ (this is the so-called ***Shabat–Yamilov lattice*** [3], which appears as the Bäcklund transformation for Landau–Lifshitz equation). Poisson brackets and (involutive) Hamiltonians for the flows (2) are of the form

$$\{u_n, v_n\} = 2h(u_n, v_n), \quad H_\pm = \sum_n \big(\frac{1}{2} \log h(u_n, v_n) - \log(u_{n\pm1} - v_n)\big).$$

The equivalence of both models is described by the following statement which makes use of the complexified stereographic projection

$$S(u, v) = \frac{1}{u - v}(1 - uv, i + iuv, u + v), \quad \langle S, S\rangle = 1.$$

Statement 1. *Let* $J = \operatorname{diag}(J_1, J_2, J_3)$, $K = \operatorname{diag}(K_1, K_2, K_3)$, $J = CI - \det K \cdot K^{-1}$ *and the polynomial* $h(u, v)$ *be*

$$h(u, v) = \frac{i}{4}(u - v)^2 \langle S(u, v), KS(u, v)\rangle.$$

The variables u_n, v_n define the vector on the sphere $S_n = S(u_n, v_n)$, then the variables

$$s^0_n = \rho\sqrt{\det K}\langle S_n, KS_n\rangle^{-1/2}, \quad s_n = -\rho\langle S_n, KS_n\rangle^{-1/2}K^{1/2}S_n$$

satisfy the Poisson brackets (1), the values of Casimir functions are equal to $c_0 = \rho^2$, $c_1 = C\rho^2$, and Hamiltonian is equal to $H = -H_+ - H_- + \mathrm{const}$.

The general linear combination of the flows (2) in the spin variables S becomes (a and b are arbitrary real constants)

$$S_{n,t} = a\langle S_n, KS_n\rangle\Big(\frac{[S_n, S_{n+1}]}{1+\langle S_n, S_{n+1}\rangle} + \frac{[S_n, S_{n-1}]}{1+\langle S_n, S_{n-1}\rangle}\Big) - 2a[S_n, KS_n] + b\langle S_n, KS_n\rangle\Big(\frac{S_n + S_{n+1}}{1+\langle S_n, S_{n+1}\rangle} - \frac{S_n + S_{n-1}}{1+\langle S_n, S_{n-1}\rangle}\Big), \qquad |S_n| = 1.$$

Sklyanin lattice corresponds to the case $b = 0$. If $K = I$ and $\rho = -1$ then variables S and s coincide. In this case the ***Heisenberg lattice*** appears

$$S_{n,t} = a\Big(\frac{[S_n, S_{n+1}]}{1+\langle S_n, S_{n+1}\rangle} + \frac{[S_n, S_{n-1}]}{1+\langle S_n, S_{n-1}\rangle}\Big) + b\Big(\frac{S_n + S_{n+1}}{1+\langle S_n, S_{n+1}\rangle} - \frac{S_n + S_{n-1}}{1+\langle S_n, S_{n-1}\rangle}\Big), \qquad |S_n| = 1. \tag{3}$$

It was introduced, at any a and b, in the paper [4], see also [5, 6] where applications in the discrete geometry were considered. It should be mentioned that the lattice corresponding to $a = 0$ remains integrable on the sphere of arbitrary dimension.

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204 Short Pulse equation

$$v_{yt} = v + \frac{1}{6}(v^3)_{yy}$$

➢ The differential substitution to sine-Gordon equation $u_{xt} = \sin u$ [5]:

$$v = u_t, \quad dy = \cos u\, dx - \frac{1}{2}u_t^2 dt.$$

➢ Zero curvature representation

$$\Psi_y = \frac{1}{2\lambda}\begin{pmatrix} 1 & v_y \\ v_y & -1 \end{pmatrix}, \quad \Psi_t = \frac{v^2}{2}U + \frac{1}{2}\begin{pmatrix} \lambda & -v \\ v & -\lambda \end{pmatrix}.$$

➢ Higher symmetry [7]:

$$v_{t_3} = \left(\frac{v_{yy}}{(1+v_y^2)^{3/2}}\right)_y$$

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205 Soliton solutions

Many nonlinear PDEs admit particular exact solutions in the form of localized travelling waves, as a rule with the amplitude depending on the velocity:

$$u(x,t) = A(k)f(kx + \omega(k)t + \delta; \alpha), \quad f(x;\alpha) \to 0, \quad x \to \pm\infty.$$

Here α denote additional parameters, such as polarization. The profile of the wave f, the amplitude A and dispersion law ω depend on the form of equation. Such solutions are called ***solitary waves***. In general, the collision of two solitary waves leads to their destruction or to appearance of small oscillations. However, several equations admit exact solutions which represent an elastic interaction of arbitrarily many solitary waves. Such solutions are called ***multi-soliton solutions*** [1].

More rigorously, the solution is called ***N-soliton*** if it is asymptotically equal to the sum of N localized travelling waves which interact without changing their shapes, amplitudes and velocities, so that the only result of interaction is the shifts of the phases and, possibly, some parameters:

$$u(x,t) \sim \sum_{i=1}^{N} A(k_i)f(k_i x + \omega(k_i)t + \delta_i^{\pm}; \alpha_i^{\pm}), \quad t \to \pm\infty.$$

There exist equations which admit 2-soliton solutions but cannot support 3-soliton one. However, in all known examples, the existence of 3-soliton solution implies the existence of N-soliton one for any N.

In some cases this definition appears too restrictive. For example, the number of solitons may changed in the 3-wave interaction.

➢ See also: KdV solitons

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206 Somos sequences

$$
\begin{aligned}
S_4:&\quad a_n a_{n-4} = a_{n-1}a_{n-3} + a_{n-2}^2, && a_0 = \cdots = a_3 = 1\\
S_5:&\quad a_n a_{n-5} = a_{n-1}a_{n-4} + a_{n-2}a_{n-3}, && a_0 = \cdots = a_4 = 1\\
S_6:&\quad a_n a_{n-6} = a_{n-1}a_{n-5} + a_{n-2}a_{n-4} + a_{n-2}^2, && a_0 = \cdots = a_5 = 1\\
S_7:&\quad a_n a_{n-7} = a_{n-1}a_{n-6} + a_{n-2}a_{n-5} + a_{n-3}a_{n-4}, && a_0 = \cdots = a_6 = 1
\end{aligned}
$$

These recurrent relations generate integers for any n. In a more general setting, a_n are Laurent polynomials on the initial data $a_0, \dots, a_{k-1}$ for S_k. This is the so-called ***Laurent property*** which is observed for some other discrete equations as well. However, it is not valid for the higher sequences S_k at $k > 7$.

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207 Squared eigenfunctions constraints

A wide and well-known class of reductions from 3-dimensional equations to vectorial 2-dimensional ones consists of so-called ***squared eigenfunction constraints***. For instance, the Manakov system [1, 2] and its third order symmetry

$$\psi_y = \psi_{xx} + 2\langle\psi,\phi\rangle\psi, \quad -\phi_y = \phi_{xx} + 2\langle\psi,\phi\rangle\phi,$$
$$\psi_t = \psi_{xxx} + 3\langle\psi,\phi\rangle\psi_x + 3\langle\psi_x,\phi\rangle\psi, \quad \phi_t = \phi_{xxx} + 3\langle\psi,\phi\rangle\phi_x + 3\langle\psi,\phi_x\rangle\phi$$

define such a reduction for the Kadomtsev–Petviashvili equation

$$4u_t = u_{xxx} - 6uu_x + 3q_y, \quad q_x = u_y$$

with respect to the quantities $u = -2\langle\psi,\phi\rangle$, $q = 2\langle\psi,\phi_x\rangle - 2\langle\psi_x,\phi\rangle$ [3]. The generic solution satisfying this reduction is determined by a pair of vector functions on x chosen as the initial data $(\psi,\phi)|_{y=0,t=0}$. Therefore, such solutions are just a special class within all solutions of KP equation which can be generically defined by arbitrary function $u|_{t=0}$ depending on two variables x, y. However, since the vector dimension can be arbitrarily large, this type of reductions of (2+1)-dimensional systems is rather important and allows one to construct rich families of exact solutions.

➢ For more examples, see Hirota–Ohta system.

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208 The symmetry approach

Author: A.B. Shabat, 28.04.2007

The classification of integrable equations is an intriguing and extremely difficult problem. For now, the exhaustive results have been obtained only for few types of equations, and the further progress requires the immense efforts. In many cases a partial classification was possible only under some additional assumptions, such as polynomiality, homogeneity, Hamiltonicity or some other special structure of the equations under scrutiny.

This article is devoted to the description of the method based on the notion of the higher symmetries which has proved to be the most effective tool for solving of the classification problems. The review of some alternative approaches is given in the article Integrability.

1. Symmetries as the test of integrability

The existence of even a single higher symmetry is the very restrictive property and may be successfully used as the test of integrability. However it may not guarantee that all found answers are really integrable. For long, the following conjecture seemed to be true.

Conjecture 1 (Fokas)**.**

➢ *If a scalar evolution equation admits one time-independent higher symmetry then it admits infinitely many [5, 6].*

➢ *For n-component systems n symmetries suffice [7].*

At $n = 2$, the first counter-example of 4-th order system which has only one higher symmetry (of 6-th order) was found by Bakirov [8]. Kamp and Sanders have proved that there exist in fact infinitely many 4-th order systems with finitely many symmetries and have found an example of 7-th order system with exactly 2 higher symmetries (of orders 11 and 29) [9].

The first part of this conjecture remains an open question till now. In any case, the classification problem based on the minimal assumption that just one higher symmetry exists is unnecessarily difficult. In order to make it more constructive, it is convenient to accept the existence of an *infinite* hierarchy of the higher symmetries as the *definition* of integrability. Technically, this leads to the following concept.

Definition 2. A scalar evolutionary equation

$$u_t = F(x, u, u_1, u_2, \dots, u_n) \tag{1}$$

with one spatial variable possesses the ***formal symmetry*** if the equation

$$D_t(A) = [F_*, A], \quad F_* := F_n D_x^n + \dots + F_1 D_x + F_0 \tag{2}$$

admits the solution $A = a_{-1}D_x + a_0 + a_1 D_x^{-1} + a_2 D_x^{-2} + \dots$ where all coefficients a_k are local functions on $x, u, u_1, \dots$.

The justification of this definition is given in the next section. In some aspects the analysis of equation (2) is analogous to the classical problem of description of commuting differential operators, see Theorem 43.4.

It turns out that equation (2) is equivalent to an infinite sequence of the obstacles to integrability, of the form

$$D_x(a_k) = \text{expression depending on } F \text{ and } a_{k-1}, \dots, a_0, a_{-1},$$

moreover, this can be rewritten in the equivalent form of conservation laws

$$D_x(\sigma_k) = D_t(\rho_k), \quad k = -1, 0, 1, \dots \tag{3}$$

where the so called ***canonical densities*** ρ_k are expressed by the certain algorithm described below through the right hand side of the equation and the previously defined σ_i. For a given equation this provides an easy to check test of integrability. It can be also used for the classification of the integrable equations of the fixed order.

2. Necessary integrability conditions

Accordingly (209.5) the compatibility condition for two evolutionary equations

$$D_{t_1}(u) = F_1(x, u, u_1, u_2, \dots, u_{n_1}), \quad D_{t_2}(u) = F_2(x, u, u_1, u_2, \dots, u_{n_2}) \tag{4}$$

can be rewritten in the form

$$[D_{t_1} - (F_1)_*, D_{t_2} - (F_2)_*] = 0 \quad \Leftrightarrow \quad D_{t_1}((F_2)_*) - D_{t_2}((F_1)_*) = [(F_1)_*, (F_2)_*]. \tag{5}$$

Definition 3. ***Integrable hierarchy, (IH)*** is the series

$$A = a_0 D + a_1 + a_2 D^{-1} + a_3 D^{-2} + \dots, \quad a_j \in \mathcal{U} \tag{6}$$

together with a set of functions

$$\mathcal{H}(A) = \{F_n \in \mathcal{U} \ : D_n(A) = D_n(a_0)D + D_n(a_1) + D_n(a_2)D^{-1} + \dots = [F_{n,*}, A]\} \tag{7}$$

where $D_n(u) := F_n$.

We call A the ***basic*** operator of the hierarchy and consider as identical one another basic operator $\tilde{A} = \alpha(D)A$ obtained by multiplication on the series

$$\alpha_0 D + \alpha_1 + \alpha_2 D^{-1} + \alpha_3 D^{-2} + \dots, \quad \alpha_j \in \mathbb{C}$$

with constant coefficients.

Clearly, for F_1, $F_2 \in \mathcal{H}(A)$ we can define $F_3 \in \mathcal{U}$ using following general formulae:

$$F_3 = D_1(F_2) - D_2(F_1) \quad \Leftrightarrow \quad F_3 = F_{2,*}(F_1) - F_{1,*}(F_2) := \{F_1, F_2\}. \tag{8}$$

Then

$$D_3 = [D_1, D_2], \quad D_3 - (F_3)_* = [D_1 - (F_1)_*, D_2 - (F_2)_*]$$

and the Jacobi identity implies that $F_3 \in \mathcal{H}(A)$. Thus $\mathcal{H}(A) \subset \mathcal{U}$ is a Lie algebra with multiplication (8). One can see $f_1 = u_1$ belongs to any hierarchy since $f_{1,*} = D$ and for any formal series (6) we have

$$[D, A] = D(a_0)D + D(a_1) + D(a_2)D^{-1} + \dots = D(A).$$

On the other hand, the compatibility condition (7) became extremely restrictive in the case of higher order $m \geq 2$ of the function $F \in \mathcal{U}$. Most known IH bear the names of the corresponding equations $u_t = F \in \mathcal{H}$

of the minimal order n_0. Here we have an analogy with the DO B of the minimal order $m > 1$ in the nontrivial centraliser $B \in \mathcal{C}(A)$ (see end of the previous section). Moreover, in virtue of Svinolupov theorem, this minimal order for "nontrivial" IH satisfies the condition $n_0 > 2$. Well known integrable hierarchies correspond to the following list of third order equations.

KdV type equations

$$u_t = u_3 + P(u)u_1, \quad P''' = 0, \tag{9}$$

$$u_t = u_3 - \frac{1}{2}u_1^3 + (\alpha e^{2u} + \beta e^{-2u})u_1, \tag{10}$$

$$u_t = u_3 - \frac{3}{2}\frac{u_2^2}{u_1} + \frac{r(u)}{u_1}, \quad r^{(5)} = 0. \tag{11}$$

For a comparison each with others distinct integrable hierarchies the following definition is useful.

Definition 4. The integrable hierarchy $\mathcal{H}(A_2)$ is called ***reducible*** to $\mathcal{H}(A_1)$ if a differential operator B exists with coefficients from $\mathcal{U}$ such that

$$A_2 = B \circ A_1 \circ B^{-1}.$$

Example 5 (Burgers-hierarchy). One can easily verify that $A_t = [f_*, A]$ in the case

$$u_t = u_{xx} + 2uu_x = f, \quad A = D + u + u_1 D^{-1}.$$

Thus, a basic operator for the Burgers equation

$$A = D + u + u_1 D^{-1} = D(D + u)D^{-1},$$

is related to $\hat{A} = D + u$ by the conjugation. One can verify that evolution equation

$$D_\tau(u) = D_x(u_2 + 3uu_x + u^3)$$

belongs to the Burgers hierarchy $\mathcal{H}(A)$ as well.

It is not difficult to prove that

$$A_t = [F_*, A] \quad \Leftrightarrow \quad (A^j)_t = [F_*, A^j], \quad j = -1, 1, 2, \dots.$$

Therefore, the order of the formal series (6) in the definition of the integrable hierarchy may be arbitrary.

Next, let $F \in \mathcal{U}$ and $\operatorname{ord} F_* = m > 1$, i.e.

$$F_* = f_0 D^m + f_1 D^{m-1} + \dots + f_m.$$

Put in the compatibility condition $A_t = [f_*, A]$

$$A = a_0 D^m + a_1 D^{m-1} + a_2 D^{m-2} + \dots$$

with undeterminate coefficients. Then collecting the coefficients by D^k, $k = 2m-1, 2m-2, \dots, m+1$ shows that in the Definition 3 one may set, without loss of generality,

$$A = F_* + g_0 D + g_1 + g_2 D^{-1} + \dots. \tag{12}$$

In other words first $m-1$ coefficients of the series (6) are related with the coefficients of the m-th order differential operator F_* if $F \in \mathcal{H}(A)$.

Example 6 (Linear equations). Hierarchies $\mathcal{H}(A)$ with differential operators A correspond to linear equations (1). In the second order case with $A = D^2 + a$ we obtain $F_j = A^j(u) \in \mathcal{H}$ and, particularly,

$$u_t = u_{xx} + a(x)u = F \quad \Leftrightarrow \quad A_t = [F_*, A].$$

For "special potentials" a when there is odd order DO $B \in \mathcal{C}(A)$ there arise additional terms $\tilde{F}_k = B^k(u)$ of this hierarchy.

Example 7 (KdV-hierarchy). In KdV case one can find the recursion operator A (cf (12)):

$$u_t = u_{xxx} + 6uu_x := F, \quad F_* = D^3 + 6uD + 6u_x, \quad A = D^2 + 4u + 2u_x D^{-1} \tag{13}$$

The check of compatibility condition $A_t = [F_*, A]$ is easy:

$$4F + 2D(F)D^{-1} = [D^3 + 6Du, D^2 + 4u + 2u_1 D^{-1}].$$

The intermediate corollary of the formula (13) and Theorem 10 below is the sequence of the local conservation laws of KdV equation i.e. differential corollaries of the equation (13) of the divergent form:

$$D_t(\rho) = D_x(\sigma), \quad \rho, \sigma \in \mathcal{U}.$$

The densities ρ of these conservation laws are common for all members of the hierarchy and are defined as follows.

Definition 8. For an integrable hierarchy $\mathcal{H}(A)$ with defining operator (6) the ***canonical series*** of the densities $\rho_j \in \mathcal{U}$, $j = -1, 1, 2, \ldots$ is as follows

$$\rho_j = \operatorname{res} A^j, \quad j = -1, 1, 2, \ldots. \tag{14}$$

Lemma 9. *For all $m, n \in \mathbb{Z}$*

$$\operatorname{res}[aD^m, bD^n] = D_x \alpha_{m,n}$$

where $\alpha_{m,n}$ is a differential polynomial on a and b.

Proof. The residue vanish if the powers m, n obey the condition $mn \geq 0$. For instance in the case $n = 0$ the commutator $aD^m b - baD^m$ is a DO if $m \geq 0$ and PDO of order $m - 1 \leq -2$ if $m < 0$. Obviously the coefficient by D^{-1} is zero in both cases. Obviously as well that the residue vanish if $m + n < 0$.

Let now m, n have different signs and $m + n = k \geq 0$. Then

$$\operatorname{res}[aD^m, bD^n] = \binom{m}{k+1}(aD^{k+1}(b) + (-1)^k D^{k+1}(a)b)$$

since

$$m + n = k \quad \Rightarrow \quad m(m-1)\cdots(m-k) = \pm n(n-1)\cdots(n-k).$$

Standard "integration by parts" completes the proof. Particularly for $k = 0, 1$ we have, respectively

$$aD(b) + D(a)b = D(ab) \quad aD^2(b) - D^2(a)b = D(aD(b) - D(a)b). \qquad \blacksquare$$

Theorem 10. *Let A be the basic operator (6) of the integrable hierarchy $\mathcal{H}(A)$. Then, for any $F \in \mathcal{H}(A)$ the equation $u_t = F$ possess a series of conservation laws with the canonical densities (14).*

Proof. Using Lemma 9 we find

$$A_t = [f_*, A] \quad \Rightarrow \quad A_t^j = [f_*, A^j] \quad \Rightarrow \quad D_t(\operatorname{res} A^j) = \operatorname{res}[f_*, A^j] \in \operatorname{Im} D.$$

In the case ρ_0 we have

$$A_t = [f_*, A] \quad \Rightarrow \quad A_t A^{-1} = F_* - AF_*A^{-1} = [A^{-1}, AF_*].$$

Therefore

$$\operatorname{res}(A_t A^{-1}) = \operatorname{res}\{(a_{0,t}D + a_{1,t} + \dots)(b_0 D^{-1} + b_1 D^{-2} + \dots)\} = \left(\frac{a_1}{a_0}\right)_t \in \operatorname{Im} D.$$

We used here the equalities $a_0 b_0 = 1$, $a_0 b_1 + a_1 b_0 = 0$. ■

Particularly, this theorem imply that the canonical density $\rho_{-1} = a_0^{-1}$ generates local conservation law for any n-th order equation $u_t = F_n \in \mathcal{H}(A)$. It follows from formulae (12) that $a_0^n = \partial F_n / \partial u_n$ and this gives rise to the ***first integrability condition***

$$D_t \left(\frac{\partial F_n}{\partial u_n}\right)^{-\frac{1}{n}} \in \operatorname{Im} D \tag{15}$$

which is necessary condition for $u_t = F_n$ to belong to some integrable hierarchy $\mathcal{H}(A)$.

We should recall that two conservation laws with the densities ρ_1 and ρ_2 are considered equivalent if $\rho_2 \sim \rho_2$ and $\rho \sim 0$ means

$$\rho = D_x(\sigma) \quad \Rightarrow \quad D_t(\rho) = D_x(D_t \sigma).$$

This conservation law is considered trivial.

Integrability conditions analogous (15) play important role in the classification of IH and equation (12) allows to rewrite several first canonical densities explicitly.

Example 11. For an evolutionary PDE (1) of the form

$$u_t = u_n + F(u, u_1, \dots, u_k), \quad k < n, \quad n \geq 2 \tag{16}$$

$F_k = \partial_{u_k} F$ is a density of a local conservation law. For equations of the third order

$$u_t = u_3 + F(x, u, u_1), \quad \rho_1 = F_1, \quad \rho_2 = F_0, \quad \rho_3 = \sigma_1 \tag{17}$$

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209 Symmetry, higher

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Due to the Bäcklund theorem the groups of transformations depending on the higher order derivatives do not exist. However, the very natural and rich in content generalization is possible of the infinitesimal definition.

In a very general sense, a symmetry of a partial differential (or difference equation) is just another equation which is consistent with it. Consider the simplest case of the evolutionary PDEs with one spatial variable

$$u_t = F(x, u, u_1, \dots, u_n) \tag{1}$$

where u_k stands for k-th order derivative with respect to x. One says that another such equation

$$u_T = G(x, u, u_1, \dots, u_m)$$

is a generalized or higher symmetry of (1) if the cross derivatives coincide: $u_{tT} = u_{Tt}$, or $D_T(F) = D_t(G)$.

To make this definition precise one have to formalize the definition of x, t- and T-derivatives. We consider $x, u, u_1, \dots$ as independent ***dynamical variables*** (this approach is traditional for the differential algebra, see e.g. [3], where u_k are called *differential indeterminate*). Then the x-derivative is replaced with the operator ***of total derivative***

$$D_x = \partial_x + u_1\partial_u + \dots + u_{k+1}\partial_{u_k} + \dots,$$
$$D_x : \quad x \to 1, \quad u \to u_1 \to \dots \to u_k \to u_{k+1} \to \dots \ .$$

Let $\mathcal{F}$ denotes the set of locally smooth functions on the finite number of dynamical variable. Any such function can be differentiated with respect to t in virtue of equation (1) accordingly to the chain rule:

$$D_t(G) = G_u F + G_{u_1} D_x(F) + \dots + G_{u_m} D_x^m(F).$$

The result can be conveniently written as

$$D_t(G) = \nabla_F(G) = G_*(F)$$

by use of the vector field called ***evolutionary derivative***

$$\nabla_F := F\partial_u + D_x(F)\partial_{u_1} + \dots + D_x^k(F)\partial_{u_k} + \dots \tag{2}$$

and the differential operator called ***Frechet derivative*** or ***Gato derivative*** or ***linearization operator***

$$G_* := G_u + G_{u_1}D_x + \dots + G_{u_m}D_x^m, \qquad G_*(v) = \Big(\frac{d}{d\varepsilon}G[u+\varepsilon v]\Big)\Big|_{\varepsilon=0}. \tag{3}$$

Definition 1. An evolutionary PDE $u_T = G(x,u,u_1,\dots,u_m)$ is called the ***symmetry*** of (1) if the corresponding evolutionary derivatives commute: $[\nabla_F, \nabla_G] = 0$. The set of all G satisfying this equation is denoted $\mathrm{Sym}(F)$.

In addition to the commutator of vector fields, we will use the brackets for denoting of the commutator of differential operators, and also for the operation

$$[F,G] := \nabla_F(G) - \nabla_G(F) = G_*(F) - F_*(G), \qquad F,G \in \mathcal{F}. \tag{4}$$

This does not lead to any misunderstanding, since the use of notation is clear from the type of operands.

Obviously, both ∇_F and F_* are linear in F. The other important properties of the introduced operations are listed below.

Statement 2. *The identities hold:*

1) $[D_x, \nabla_F] = 0;$
2) $[\nabla_F, \nabla_G] = \nabla_{[F,G]};$
3) $[F,[G,H]] + [G,[H,F]] + [H,[F,G]] = 0;$
4) $(FG)_* = FG_* + GF_*;$
5) $(D_x(F))_* = D_xF_*;$
6) $[\nabla_F - F_*, \nabla_G - G_*] = \nabla_{[F,G]} - [F,G]_*.$

Proof. 1,2) It is sufficient to apply the commutator to the dynamical variables $x, u = u_0, u_1, \dots$ We have:

$$1)\quad [D_x, \nabla_F](x) = 0, \quad [D_x, \nabla_F](u_k) = D_x(D_x^k(F)) - \nabla_F(u_{k+1}) = 0 \quad \Rightarrow \quad [D_x, \nabla_F] = 0;$$

$$2)\quad [\nabla_F, \nabla_G](x) = 0, \quad [\nabla_F, \nabla_G](u_k) = \nabla_F(D_x^k(G)) - \nabla_G(D_x^k(F))$$
$$\stackrel{1)}{=} D_x^k(\nabla_F(G) - \nabla_G(F)) = D_x^k([F, G]) \quad \Rightarrow \quad [\nabla_F, \nabla_G] = \nabla_{[F,G]}.$$

The identity 3) follows from 2) and the Jacobi identity for the vector fields. The identity 4) is obvious, 5) requires some calculations:

$$(D_x(F))_* = \Big(\sum_k F_{u_k} u_{k+1}\Big)_* = \sum_k F_{u_k} D_x^{k+1} + \sum_{j,k} u_{k+1} F_{u_k,u_j} D_x^j = D_x \sum_k F_{u_k} D_x^k = D_x F_*.$$

In order to prove 6), we first prove the relation $(\nabla_F(G))_* = [\nabla_F, G_*] + G_* F_*$:

$$(\nabla_F(G))_* = \Big(\sum_k G_{u_k} D_x^k(F)\Big)_* = \sum_k \Big(G_{u_k} D_x^k F_* + D_x^k(F) \sum_j G_{u_k,u_j} D_x^j\Big)$$
$$= G_* F_* + \sum_{j,k} D_x^k(F) G_{u_k,u_j} D_x^j = G_* F_* + \sum_j \nabla_F(G_{u_j}) D_x^j = G_* F_* + [\nabla_F, G_*].$$

Now, 6) follows from here and 2):

$$[\nabla_F - F_*, \nabla_G - G_*] = [\nabla_F, \nabla_G] - [\nabla_F, G_*] - G_* F_* + [\nabla_G, F_*] + F_* G_*$$
$$= \nabla_{[F,G]} - (\nabla_F(G))_* + (\nabla_G(F))_* = \nabla_{[F,G]} + [F, G]_*. \qquad \blacksquare$$

In particular, this statement implies that the following definitions of symmetry are equivalent:

$$[\nabla_F, \nabla_G] = 0 \quad \Leftrightarrow \quad [F, G] = 0 \quad \Leftrightarrow \quad (\nabla_F - F_*)(G) = 0 \quad \Leftrightarrow \quad [\nabla_F - F_*, \nabla_G - G_*] = 0. \qquad (5)$$

The latter formula is of particular importance for the definition of the necessary integrability conditions (see symmetry approach).

Moreover, the Jacobi identity 3) implies that the space $\mathcal{F}$ equipped with the operation $[,]$ is a Lie algebra and the symmetries set $\mathrm{Sym}(F)$ is its Lie subalgebra. The ***integrable equations*** can be defined as those with an infinite-dimensional Lie algebra of symmetries. The structure of this Lie algebra can be different. If the equation is linearizable then it contains an infinite-dimensional Lie subalgebra of classical symmetries. For the KdV-type equations it is typical that classical symmetries form a finite-dimensional noncommutative Lie subalgebra while the higher symmetries form an infinite-dimensional commutative Lie subalgebra (called "hierarchy").

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210 Thomas equation

$$u_{xt} = au_t + bu_x + cu_xu_t$$

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211 Toda lattice

$$q_{n,xx} = e^{q_{n+1}-q_n} - e^{q_n-q_{n-1}}$$

➢ The higher symmetries

$$q_{n,t_2} = q_{n,x}^2 + e^{q_{n+1}-q_n} + e^{q_n-q_{n-1}}, \qquad q_{n,t_3} = q_{n,x}^3 + (q_{n+1}+2q_n)_x e^{q_{n+1}-q_n} + (2q_n+q_{n-1})_x e^{q_n-q_{n-1}}, \quad \dots$$

can be rewritten as the NLS hierarchy for the variables $u = e^{q_n}$, $v = e^{-q_{n-1}}$:

$$u_{t_2} = u_{xx} + 2u^2v, \quad -v_{t_2} = v_{xx} + 2v^2u; \qquad u_{t_3} = u_{xxx} + 6uvu_x, \quad v_{t_3} = v_{xxx} + 6uvv_x; \quad \dots$$

➢ Hamiltonian structure ($p_n = q_{n,x}$):

$$\{p_n, q_n\} = 1, \quad H = \sum\Big(\frac{1}{2}p_n^2 + e^{q_{n+1}-q_n}\Big).$$

➢ Zero curvature representation:

$$L_n = \begin{pmatrix} q_{n,x}+2\lambda & e^{q_n} \\ -e^{-q_n} & 0 \end{pmatrix}, \quad U_n = \begin{pmatrix} -\lambda & -e^{q_n} \\ e^{-q_{n-1}} & \lambda \end{pmatrix}$$

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212 Toda lattice, two-dimensional

$$q_{n,xy} = e^{q_{n+1}-q_n} - e^{q_n-q_{n-1}}$$

➣ Introduced in [1]

➣ In slightly different form, 2D Toda lattice describes the sequence of Laplace invariants.

➣ Infinite Toda lattice admits a number of reductions to finite exponential systems. The most important examples are related to semisimple Lie algebras, see e.g. [2].

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213 Toda lattice, relativistic

$$q_{n,xx} = g^2 q_x \left(q_{n-1,x} \frac{e^{q_{n-1}-q_n}}{1+g^2 e^{q_{n-1}-q_n}} - q_{n+1,x} \frac{e^{q_n-q_{n+1}}}{1+g^2 e^{q_n-q_{n+1}}} \right)$$

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214 Tops. Pairs of commuting Hamiltonians quadratic in momenta

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In this article we study the pairs of Hamiltonians of the form

$$H = ap_1^2 + 2bp_1p_2 + cp_2^2 + dp_1 + ep_2 + f, \quad K = Ap_1^2 + 2Bp_1p_2 + Cp_2^2 + Dp_1 + Ep_2 + F \tag{1}$$

commuting with respect to the standard Poisson bracket $\{p_\alpha, q_\beta\} = \delta_{\alpha\beta}$. The coefficients of the Hamiltonians are assumed to be locally analytical functions of q_1, q_2. This problem was considered in papers [1, 2, 3, 4, 5, 6, 7]. Here we present some new many-parametrical families of such pairs and the universal method of constructing the full solution of Hamilton–Jacobi equation in terms of integrals on some algebraic curve. In several examples this curve is a non-hyperelliptic covering over an elliptic curve.

1. Diagonalization of quadratic part

It is possible to introduce new coordinates s_1, s_2 such that quadratic parts of H, K became diagonal. The condition $\{H, K\} = 0$ is essential for existence of this transformation. Let s_1, s_2 be roots of equation

$$\Phi(s, q_1, q_2) = (B - bs)^2 - (A - as)(C - cs) = 0$$

and $\Phi^i = \Phi(s_i, q_1, q_2)$, then the canonical transformation

$$(q_1, q_2, p_1, p_2) \to (s_1, s_2, P_1, P_2) : p_1 = -\left(\frac{\Phi^1_{q_1}}{\Phi^1_{s_1}} P_1 + \frac{\Phi^2_{q_1}}{\Phi^2_{s_2}} P_2\right), \quad p_2 = -\left(\frac{\Phi^1_{q_2}}{\Phi^1_{s_1}} P_1 + \frac{\Phi^2_{q_2}}{\Phi^2_{s_2}} P_2\right)$$

brings the pair (1) to the form

$$H = \frac{S_1(s_1)}{s_1 - s_2} P_1^2 - \frac{S_2(s_2)}{s_1 - s_2} P_2^2 + \tilde{d} P_1 + \tilde{e} P_2 + \tilde{f}, \qquad K = \frac{s_2 S_1(s_1)}{s_1 - s_2} P_1^2 - \frac{s_1 S_2(s_2)}{s_1 - s_2} P_2^2 + \tilde{D} P_1 + \tilde{E} P_2 + \tilde{F},$$

where

$$S_i(s_i) = \frac{1}{(\Phi^i_{q_i})^2}((as_i - A)(\Phi^i_{q_1})^2 + 2(bs_i - B)\Phi^i_{q_1}\Phi^i_{q_2} + (cs_i - C)(\Phi^i_{q_2})^2).$$

Theorem 1. *Any pair of commuting Hamiltonians (1) can be brought by canonic transformation*

$$\hat{P}_1 = P_1 + \frac{\partial F(s_1, s_2)}{\partial s_1}, \qquad \hat{P}_2 = P_2 + \frac{\partial F(s_1, s_2)}{\partial s_2}$$

to the pair of the form

$$H = \frac{U_1 - U_2}{s_1 - s_2}, \qquad K = \frac{s_2 U_1 - s_1 U_2}{s_1 - s_2} \tag{2}$$

where

$$\begin{aligned} U_1 &= S_1(s_1)P_1^2 + \frac{\sqrt{S_1(s_1)S_2(s_2)}Z_{s_1}}{(s_1 - s_2)}P_2 - \frac{S_1(s_1)Z_{s_1}^2}{4(s_1 - s_2)^2} + V_1(s_1, s_2), \\ U_2 &= S_2(s_2)P_2^2 - \frac{\sqrt{S_1(s_1)S_2(s_2)}Z_{s_2}}{(s_1 - s_2)}P_1 - \frac{S_2(s_2)Z_{s_2}^2}{4(s_2 - s_1)^2} + V_2(s_1, s_2), \end{aligned} \tag{3}$$

$$\begin{aligned} V_1 &= \frac{1}{2}\sqrt{S_1(s_1)}\partial_{s_1}\left(\sqrt{S_1(s_1)}\frac{Z_{s_1}^2}{s_1 - s_2}\right) + f_1(s_1), \\ V_2 &= \frac{1}{2}\sqrt{S_2(s_2)}\partial_{s_2}\left(\sqrt{S_2(s_2)}\frac{Z_{s_2}^2}{s_2 - s_1}\right) + f_2(s_2) \end{aligned} \tag{4}$$

for some functions $Z(s_1, s_2)$, $S_i(s_i)$ and $f_i(s_i)$. The Poisson bracket $\{H, K\}$ equals to zero if and only if the following conditions are fulfilled:

$$Z_{s_1,s_2} = \frac{Z_{s_1} - Z_{s_2}}{2(s_2 - s_1)}, \tag{5}$$

$$(Z_{s_1}\partial_{s_2} - Z_{s_2}\partial_{s_1})\left(\frac{V_1 - V_2}{s_1 - s_2}\right) = 0. \tag{6}$$

The general analytical solution of Euler–Darboux equation (5) has the following expansion in the neighborhood of the singular line $x = y$:

$$Z(x,y) = A + \log(x-y)B, \quad A = \sum_0^\infty a_i(x+y)(x-y)^{2i}, \quad B = \sum_0^\infty b_i(x+y)(x-y)^{2i}.$$

Here a_0 and a_1 are arbitrary functions and the other coefficients are expressed through these two functions and their derivatives. For example, $b_0 = \frac{1}{2}a_0''$.

We insert this expansion into (6) in order to prove $B = 0$. It is easy to check that any solution of the equation (5) with $B = 0$ is of the form

$$Z(x,y) = z_0 + \delta(x+y) + (x-y)^2 \sum_{k=0}^\infty \frac{g^{(2k)}(x+y)}{2^{(2k)}k!(k+1)!}(x-y)^{2k}, \tag{7}$$

where $g(x)$ is some function and z_0, δ are constants. We call $g(x)$ *generating function* for (7). Without loss of generality we choose $z_0 = 0$. The parameter δ is very important for classification of Hamiltonians from Theorem 1.

We find all functions Z, corresponding to the rational generating functions g. Choosing $g(x) = x^n$, we obtain an infinite set of polynomial solutions $Z^{(n)}$ for (5). In particular,

$$\begin{array}{lcl} g(x) = 1 & \Leftrightarrow & Z^{(0)}(x,y) = (x-y)^2, \\ g(x) = x & \Leftrightarrow & Z^{(1)}(x,y) = (x+y)(x-y)^2, \\ g(x) = x^2 & \Leftrightarrow & Z^{(2)}(x,y) = \frac{1}{4}\left((x-y)^2 + 4(x+y)^2\right)(x-y)^2. \end{array}$$

The whole set can be obtained by applying 'creating' operator $x^2\partial_x + y^2\partial_y - \frac{1}{2}(x+y)$ to Z^0. The rational functions $g(x) = (x-\mu)^{-n}$ correspond to another class of exact solution of equation (5), for example

$$g_\mu(x) = \frac{1}{4(x-2\mu)} \quad \Leftrightarrow \quad Z_\mu(x,y) = \sqrt{(\mu-x)(\mu-y)} + \frac{1}{2}(x+y) - \mu.$$

The solution corresponding to the poles of order $n \geq 2$ can be obtained by differentiating the last formula with respect to the parameter μ. Since function Z is linear in g we obtained the solution Z with rational generating function $g(x) = \sum_i c_i x^i + \sum_{i,j} d_{ij}(x-\mu_i)^{-j}$.

Conjecture 2. *For all Hamiltonians (2)–(6) the generating function g is rational of the form $g(x) = \frac{P(x)}{S(x)}$, where P and S are polynomials with* $\deg P < 5$, $\deg S < 6$.

In papers [5, 6] the following solution of the system (5), (6) has been considered:

$$Z(x,y) = x + y, \quad S_1(x) = S_2(x) = \sum_{i=0}^{6} c_i x^i, \quad f_1(x) = f_2(x) = -\frac{3}{4}c_6 x^4 - \frac{1}{2}c_5 x^3 + \sum_{i=0}^{2} k_i x^i$$

where c_i, k_i are constants. It should be noted that Clebsch top and $so(4)$-Schottky–Manakov top [8, 9, 10] are the particular cases of this model [6]. The full solution of Hamilton–Jacobi equation of this model was obtained in [6] by means of some kind of separation of variables on a non-hyperelliptic curve of genus 4.

2. Universal solution of Hamilton–Jacobi equation

Let H and K be of the form (2)–(4) and let $p_1 = F_1(x,y)$, $p_2 = F_2(x,y)$ be solution of the system $H = e_1$, $K = e_2$ where e_i are constants. Here and below we denote for short $x = s_1$, $y = s_2$. Accordingly to Jacobi lemma, if $\{H, K\} = 0$ then $F_{1,y} = F_{2,x}$. To find the action $\mathsf{S}(x, y, e_1, e_2)$ it is sufficient to solve the system

$$\mathsf{S}_x = F_1, \quad \mathsf{S}_y = F_2.$$

We rewrite the system $H = e_1$, $K = e_2$ in the form

$$p_1^2 + ap_2 + b = 0, \qquad p_2^2 + Ap_1 + B = 0, \tag{8}$$

where

$$a = \frac{Z_x}{x-y}\sqrt{\frac{S_2(y)}{S_1(x)}}, \qquad A = -\frac{Z_y}{x-y}\sqrt{\frac{S_1(x)}{S_2(y)}}, \tag{9}$$

$$b = -\frac{Z_x^2}{4(x-y)^2} + \frac{V_1 - e_1 x + e_2}{S_1(x)}, \qquad B = -\frac{Z_y^2}{4(x-y)^2} + \frac{V_2 - e_1 y + e_2}{S_2(y)}. \tag{10}$$

It easy to prove the following identities (the last one is obtained by use of (5), (6)):

$$2b_y + Aa_x + 2aA_x = 0, \quad 2Aa_y + aA_y + 2B_x = 0, \tag{11}$$

$$Ab_x - aB_y + 2A_x b - 2a_y B = 0. \tag{12}$$

Using the standard technique of Lagrange resolvents, we rewrite the system (8) in the form

$$uv = \frac{1}{4}aA, \quad Au^3 + \frac{4b}{a}u^2 v - \frac{4B}{A}uv^2 - av^3 = 0, \tag{13}$$

which is equivalent to a cubic equation on u^2. Let (u_k, v_k), $k = 1, 2, 3$ be solutions of (13) then

$$u_1^2 + u_2^2 + u_3^2 = -b, \quad v_1^2 + v_2^2 + v_3^2 = -B, \quad 8u_1u_2u_3 = -a^2 A, \quad 8v_1v_2v_3 = -A^2 a$$

and the formulas

$$\begin{aligned} p_1 &= u_1 + u_2 + u_3, \quad p_2 = v_1 + v_2 + v_3; \\ p_1 &= u_3 - u_1 - u_2, \quad p_2 = v_3 - v_1 - v_2; \\ p_1 &= u_2 - u_1 - u_3, \quad p_2 = v_2 - v_1 - v_3; \\ p_1 &= u_1 - u_2 - u_3, \quad p_2 = v_1 - v_2 - v_3 \end{aligned}$$

define four solutions of (8). Consider first of them.

Lemma 3. *The equations $u_{i,y} = v_{i,x}$ hold for $i = 1, 2, 3$.*

Proof. Differentiating equations (13) with respect to x and y we find u_y and v_x as functions on u and v. Then expressing v through u we obtain that $u_y = v_x$ is equivalent to identities (11), (12). ■

The above Lemma means that variables u_1, u_2, u_3 are "particular" separation variables. Indeed, the action takes the form $\mathsf{S} = \mathsf{S}_1 + \mathsf{S}_2 + \mathsf{S}_3$ where functions S_i are defined from the system

$$\mathsf{S}_{i,x} = u_i, \quad \mathsf{S}_{i,y} = v_i.$$

Let

$$u = \frac{Z_x}{2(x-y)}\sqrt{\frac{y-\xi}{x-\xi}}, \qquad v = -\frac{Z_y}{2(x-y)}\sqrt{\frac{x-\xi}{y-\xi}}.$$

It easy to see that the pair (u, v) is a solution of (13) for all ξ. If Z is a solution of (5) then $u_y = v_x$. Using this fact we introduce the function $\sigma(x, y, \xi)$ such that $\sigma_x = u$, $\sigma_y = v$. In the case of a rational function g the corresponding function Z is expressed through quadratic radicals and the function σ can be found explicitly.

After multiplication of second equation (13) by expression

$$-2\frac{\sqrt{S_1(x)}\sqrt{S_2(y)}\sqrt{x-\xi}\sqrt{y-\xi}\,(x-y)}{Z_x Z_y}$$

it takes the form

$$\Psi(x, y, \xi) = -e_2 + e_1\xi + \frac{y-\xi}{x-y}\Big(V_1 - \frac{S_1(x)Z_x^2}{4(x-\xi)(x-y)}\Big) - \frac{x-\xi}{x-y}\Big(V_2 + \frac{S_2(y)Z_y^2}{4(y-\xi)(x-y)}\Big) = 0.$$

Statement 4. *Let the conditions (5), (6) hold. Then the function $\Psi(x, y, \xi)$ depends on the variables $Y = \sigma_\xi$ and ξ only: $\Psi(x, y, \xi) = \phi(\xi, Y)$.*

Proof. Consider Jacobian $J = \Psi_x Y_y - \Psi_y Y_x$. We change Y_y, Y_x to v_ξ, u_ξ respectively, then Jacobian J vanishes identically in virtue of (5), (6). The function ϕ can be found by setting $y = x$. ■

Equation $\phi(\xi, Y) = 0$ defines a curve, and differentials of this curve define the function of action S. Let $\xi_k(x, y)$, $k = 1, 2, 3$ be the roots of the cubic equation $\Psi(x, y, \xi) = 0$.

Theorem 5. *Function of action S is of the form*

$$\mathsf{S}(x, y) = \sum_{k=1}^{3}\Big(\sigma(x, y, \xi_k) - \int^{\xi_k} Y(\xi)d\xi\Big), \tag{14}$$

where $Y(\xi)$ is algebraic function defined by equation $\phi(\xi, Y) = 0$.

Proof. We obtain

$$\mathsf{S}_x(x, y) = \sum_{k=1}^{3}\sigma_x(x, y, \xi_k) + \sum_{k=1}^{3}(\sigma_\xi(x, y, \xi_k) - Y(\xi_k))\xi_{k,x} = \sum_{k=1}^{3} u_k = p_1.$$

Analogously, $\mathsf{S}_y(x, y) = p_2$. ■

3. Examples

Here we list all known at the moment pairs of Hamiltonians (2)–(6).

Class 1. For models of this class

$$S_1 = S_2 = S, \qquad f_1 = f_2 = f. \tag{15}$$

Theorem 6. *Let*

$$g = \frac{\tilde{G}}{S}, \quad \tilde{G} = G - \frac{\delta}{10} S', \quad f = -\frac{4\tilde{G}^2}{S} - \frac{4\delta}{3}\tilde{G}' - \frac{\delta^2}{12} S'',$$

where $S(x) = s_5x^5 + s_4x^4 + s_3x^3 + s_2x^2 + s_1x + s_0$, $G(x) = g_3x^3 + g_2x^2 + g_1x + g_0$ *and* s_i, g_i, δ *are constants. Then functions* S, f *and function* Z *corresponding to the generating function* g *(see Section 214) satisfy the systems (5), (6).*

Remark 7. The parameter δ in Theorem 6 coincides with parameter in (7). In the case $\delta = 0$ this Theorem describes all pairs of Hamiltonians (2)–(6), (15).

Consider a general case

$$S(x) = s_5(x - \mu_1)(x - \mu_2)(x - \mu_3)(x - \mu_4)(x - \mu_5),$$

where $s_5 \neq 0$ and all zeroes μ_i are distinct. Then the function Z is of the form

$$Z(x, y) = \sum_{i=1}^{5} \nu_i \sqrt{(\mu_i - x)(\mu_i - y)}, \quad nu_i = \text{const}\,. \tag{16}$$

Coefficients g_i and δ are expressed through constants ν_i from (7). For example, $2\delta = -\sum \nu_i$. Function f is defined by

$$f(x) = -\frac{1}{16} \sum_{i=1}^{5} \nu_i^2 \frac{S'(\mu_i)}{x - \mu_i} + k_1 x + k_0,$$

with constant k_1, k_0.

Calculation for a function (16) gives

$$\sigma(x,y,\xi) = -\frac{1}{2}\sum_{i=1}^{5}\nu_i \log\frac{\sqrt{x-\xi}\sqrt{y-\mu_i}+\sqrt{y-\xi}\sqrt{x-\mu_i}}{\sqrt{x-y}\sqrt{\mu_i-\xi}}, \tag{17}$$

$$Y = \frac{1}{4}\sum_{i=1}^{N}\nu_i\frac{\sqrt{(x-\mu_i)(y-\mu_i)}}{(\xi-\mu_i)\sqrt{(x-\xi)(y-\xi)}}.$$

Algebraic curve is hyperelliptic of genus 2: $\phi(Y,\xi) = S(\xi)Y^2 + f(\xi) - \xi e_1 + e_2 = 0$.

Steklov top on $so(4)$ [11] is a particular case of Theorem 6.

Class 2. Functions Z for models of this class are the special cases of the functions Z of Class 1. But this Class contains much more parameters than in Theorem 6.

These functions Z can be defined as solutions of system

$$Z_{xy} = \frac{Z_x - Z_y}{2(y-x)} = \frac{1}{3}U(Z)Z_xZ_y, \tag{18}$$

where U are some functions of one variable.

Remark 8. It easy to see that this class of solutions of Euler–Darboux equation $Z_{xy} = \frac{Z_x - Z_y}{2(y-x)}$ coincide with class of solutions of the form

$$Z = F\left(\frac{h(x)-h(y)}{x-y}\right),$$

where $U = F''/F'^2$.

Lemma 9. *The system (18) is compatible if and only if*

$$U = \frac{3B'}{2B}, \quad B(Z) = b_2Z^2 + b_1Z + b_0, \quad b_i = \text{const}.$$

Three cases are possible:

$$\deg B = 2: \quad Z = \sqrt{(x-\mu_1)(y-\mu_1)} + \sqrt{(x-\mu_2)(y-\mu_2)}, \qquad b_2 = 1, \quad b_1 = 0, \quad b_0 = -(\mu_1-\mu_2)^2, \quad (19)$$

$$\deg B = 1: \quad Z = \sqrt{x\,y} + \frac{1}{2}(x+y), \qquad b_1 = 1, \quad b_2 = b_0 = 0, \quad (20)$$

$$\deg B = 0: \quad Z = x + y. \quad (21)$$

Case $\deg B = 2$. Consider function Z of the form (19). Then

$$S(x) = (x-\mu_1)(x-\mu_2)P(x) + (x-\mu_1)^{3/2}(x-\mu_2)^{3/2}Q(x), \quad \deg P \le 3, \quad \deg Q \le 2,$$

$$f(x) = f_0 + f_1 x + k_2(x-\mu_1)^{1/2}(x-\mu_2)^{1/2} + \frac{(\mu_2-\mu_1)}{16}\Big(\frac{P(\mu_1)}{x-\mu_1} - \frac{P(\mu_2)}{x-\mu_2}\Big)$$
$$+\frac{(\mu_2-\mu_1)}{32}(x-\mu_1)^{1/2}(x-\mu_2)^{1/2}\Big(\frac{Q(\mu_1)}{x-\mu_1} - \frac{Q(\mu_2)}{x-\mu_2}\Big).$$

In the case $Q = 0$, $k_2 = 0$ these formulas coincide with the corresponding formulas of Class 1. The functions σ, Y are defined by the same formula (17) as for Class 1:

$$\sigma(x,y,\xi) = -\frac{1}{2}\sum_{i=1}^{2}\log\frac{\sqrt{x-\xi}\sqrt{y-\mu_i} + \sqrt{y-\xi}\sqrt{x-\mu_i}}{\sqrt{x-y}\sqrt{\mu_i-\xi}}, \quad Y = \frac{1}{4}\sum_{i=1}^{2}\frac{\sqrt{(x-\mu_i)(y-\mu_i)}}{(\xi-\mu_i)\sqrt{(x-\xi)(y-\xi)}}.$$

Algebraic curve in this case is of the form

$$(S_R(\xi) + \eta S_I(\xi))Y^2 - k_R(\xi) - \eta k_I(\xi) = 0 \quad (22)$$

where

$$S_R(x) = (x-\mu_1)(x-\mu_2)P(x), \quad S_I(x) = (x-\mu_1)(x-\mu_2)Q(x), \quad (23)$$

$$k_R(x) = -e_2 + e_1 x - f_0 - f_1 x - \frac{(\mu_2-\mu_1)}{16}\Big(\frac{P(\mu_1)}{x-\mu_1} - \frac{P(\mu_2)}{x-\mu_2}\Big), \quad (24)$$

$$k_I(x) = k_2 - \frac{1}{32}(\mu_1 - \mu_2)^2 - \frac{1}{16}(\mu_1 - \mu_2)\Big(\frac{Q(\mu_1)}{x - \mu_1} - \frac{Q(\mu_2)}{x - \mu_2}\Big), \tag{25}$$

$$\frac{1}{\eta} = \frac{1}{\sqrt{\xi - \mu_1}\sqrt{\xi - \mu_2}}\sqrt{1 - \frac{(\mu_1 - \mu_2)^2}{16(\xi - \mu_1)^2(\xi - \mu_2)^2 Y^2}}. \tag{26}$$

Expressing Y as a function of (ξ, η) and substituting to (22) we obtain 10-parameter cubic in (ξ, η) variables. So, in general the curve $\phi(Y, \xi) = 0$ is a covering over an elliptic curve. We obtain

$$\eta = \frac{\xi - \mu_1}{\frac{\sqrt{x-\mu_1}}{\sqrt{x-\mu_2}} + \frac{\sqrt{y-\mu_1}}{\sqrt{y-\mu_2}}} + \frac{\xi - \mu_2}{\frac{\sqrt{x-\mu_2}}{\sqrt{x-\mu_1}} + \frac{\sqrt{y-\mu_2}}{\sqrt{y-\mu_1}}},$$

therefore points $(\xi_1, \eta_1), (\xi_2, \eta_2), (\xi_3, \eta_3)$ lie on a straight line.

Case $\deg B = 1$. For the function Z of the form (20) we have

$$S(x) = xP(x) + x^{3/2}Q(x), \qquad \deg P \le 3, \quad \deg Q \le 2,$$

$$f(x) = -\frac{1}{16x}P(x) - \frac{1}{32\sqrt{x}}Q(x) + f_1 x + f_q\sqrt{x} + f_0, \quad Y = \frac{\xi + \sqrt{x}\sqrt{y}}{4\xi\sqrt{x - \xi}\sqrt{y - \xi}}.$$

The curve in this case can be written in the form (22) where

$$S_R(x) = xP(x), \quad S_I(x) = xQ(x),$$

$$k_R(x) = -e_2 + e_1 x - f_0 - f_1 x + \frac{1}{16x}P(x), \quad k_I(x) = \frac{1}{16x}Q(x) - f_q, \quad \eta = \frac{4Y\xi^{3/2}}{\sqrt{16Y^2\xi^2 - 1}}.$$

In (ξ, η) variables it also has the form of arbitrary cubic. Formula $\eta = \frac{\xi + \sqrt{xy}}{\sqrt{x} + \sqrt{y}}$ proves that points $(\xi_1, \eta_1), (\xi_2, \eta_2)$, ($\xi$
are collinear.

Case $\deg B = 0$. For the function Z given by (21) we have

$$S(x) = s_6 x^6 + s_5 x^5 + s_4 x^4 + s_3 x^3 + s_2 x^2 + s_1 x + s_0,$$

$$f(x)=-\frac{1}{40}S''(x)-\frac{1}{32\sqrt{x}}Q(x)+f_2x^2+f_1x+f_0,\quad Y=\frac{1}{2\sqrt{x-\xi}\sqrt{y-\xi}}.$$

The algebraic curve is

$$S(\xi)Y^6-F(\xi)Y^4-\Big(\frac{1}{8}F''(\xi)+\frac{7}{1920}S^{IV}(\xi)-\frac{k_2}{2}\Big)Y^2-\frac{s_6}{64}=0,\quad F(\xi)=-e_2+e_1\xi-f(\xi).$$

It is an arbitrary cubic in the variables (ξ,η), where $\eta=\xi^2-1/(4Y^2)$. Since $\eta=\xi(x+y)-xy$ the points $(\xi_1,\eta_1),(\xi_2,\eta_2),(\xi_3,\eta_3)$ are collinear.

Class 3. We will say that Hamiltonian (2)–(6) is non-symmetrical if $S_1(x)\neq S_2(x)$ or $f_1(x)\neq f_2(x)$.

Theorem 10 ([7]). *In non-symmetrical case the functions Z, S_i, f_i are solutions of (5), (6) if and only if*

$$\delta=0,\quad g=\frac{1}{H},\quad S_{1,2}=WH\pm MH^{3/2},\quad f_{1,2}=-\frac{4W}{H}\mp 2MH^{-1/2}\pm aH^{1/2},$$

where g is the generating function for Z and

$$W(x)=w_3x^3+w_2x^2+w_1x+w_0,\quad H(x)=h_2x^2+h_1x+h_0,\quad M(x)=m_2x^2+m_1x+m_0$$

with constant w_i,h_i,m_i,a.

Consider the general case $H(x)=(x-\mu_1)(x-\mu_2)$. The algebraic curve $\Psi(\xi,Y)=0$ in this case is of the form

$$-e_2+e_1\xi-\frac{RW(\xi)}{2(\xi-\mu_1)(\xi-\mu_2)(\mu_2-\mu_1)}+4M(\xi)\sqrt{2}Y\frac{\sqrt{\xi-\mu_1}\sqrt{\xi-\mu_2}}{(\mu_2-\mu_1)^{3/2}}\sqrt{R}+8b\sqrt{2}Y\frac{(\xi-\mu_1)^{3/2}(\xi-\mu_2)^{3/2}}{\sqrt{R}\sqrt{\mu_2-\mu_1}}=0$$

where

$$Y=\frac{\sqrt{(x-\mu_1)(y-\mu_1)}}{(\xi-\mu_1)\sqrt{(x-\xi)(y-\xi)}}-\frac{\sqrt{(x-\mu_2)(y-\mu_2)}}{(\xi-\mu_2)\sqrt{(x-\xi)(y-\xi)}},\quad R=16(\xi-\mu_1)^2(\xi-\mu_2)^2Y^2-(\mu_1-\mu_2)^2.$$

Substituting

$$Y = \frac{(\mu_1 - \mu_2)^{3/2}\eta}{4(\xi - \mu_2)(\xi - \mu_1)\sqrt{\eta^2(\mu_2 - \mu_1) - 8(\xi - \mu_1)(\xi - \mu_2)}}$$

into this equation we obtain the cubic in variables (ξ, η) with a full set of ten independent parameters. It easy to see that $\eta = a(x, y)\xi + b(x, y)$ where a, b are some functions.

Summing up, we have seen that in all cases of Classes 2 and 3 the algebraic curve is a non-hyperelliptic covering over an elliptic curve. The dynamics of the points $(\xi_1, Y_1), (\xi_2, Y_2), (\xi_3, Y_3)$ on this curve (see Theorem 5) satisfies the following condition: the projections of these points onto the elliptic base $(\xi_1, \eta_1), (\xi_2, \eta_2), (\xi_3, \eta_3)$ lie on a straight line.

Conjecture 11. *Any pair of the Hamiltonians (2)–(6) belongs to one of above three classes.*

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215 Tzitzeica equation

$$u_{xy} = e^{2u} - e^{-u}$$

Aliases: Bullough–Dodd equation [2], Zhiber–Shabat equation [3]

This equation is the reduction $v = 0$ of the system

$$u_{xy} = e^{2u} - \cosh(3v)e^{-u}, \quad v_{xy} = \sin^3(v)e^{-u}$$

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216 Variational derivative

The variational problem is called the problem on finding the extrema of the functional

$$\mathcal{L}[u] = \int_\Omega L[x, u_\sigma]dx, \quad L \in \mathbb{R}, \quad x \in \mathbb{R}^m, \quad u \in \mathbb{R}^n$$

where the ***Lagrange function*** L depends on x and a finite set of derivatives $u_\sigma = D^\sigma(u) = D_{x_1}^{\sigma_1} \cdots D_{x_m}^{\sigma_m}(u^1, \dots, u$ The solution of this problem is defined by ***Euler–Lagrange equation*** (with appropriate boundary conditions)

$$\delta L = 0, \qquad \delta = \Big(\frac{\delta}{\delta u^1}, \dots, \frac{\delta}{\delta u^n}\Big), \qquad \frac{\delta}{\delta u^j} = \sum_\sigma (-D)^\sigma \frac{\partial}{\partial u^j_\sigma} = \sum_\sigma (-D_{x_1})^{\sigma_1} \dots (-D_{x_m})^{\sigma_m} \frac{\partial}{\partial u^j_\sigma}.$$

The operator $\delta/\delta u^j$ is called the ***variational derivative***. The term ***Euler operator*** and the notation E_{u^j} are used as well. The Euler–Lagrange equation is written compactly as $L_*^{\intercal}(1) = 0$ by use of Frechet derivative and the formal conjugation of differential operators.

In the discrete setup the variational derivative is of the form

$$\frac{\delta}{\delta u^j} = \sum_\sigma T^{-\sigma} \frac{\partial}{\partial u^j_\sigma} = \sum_\sigma T_1^{-\sigma_1} \dots T_m^{-\sigma_m} \frac{\partial}{\partial u^j_\sigma}$$

where T_i is the shift operator $x_i \to x_i + 1$. The definition in the case of mixed continuous and discrete independent variables is straightforward as well.

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217 Vector field

A ***vector field*** on a manifold M is a smooth mapping $F : x \to T_xM$, $x \in M$. In a local coordinates $x = (x^1, \ldots, x^n)$ on M, the vector field is given by an expression of the form

$$F = f^1(x)\partial_{x^1} + \ldots f^n(x)\partial_{x^n}$$

where f^i are smooth functions on M and $\partial_{x^i} \in T_xM$ is the tangent vector to the i-th coordinate line.

The formula $F(a) = f^1 a_{x^1} + \ldots f^n a_{x^n}$ associates the vector field with the differentiation in the associative algebra of the smooth functions on M. The converse is true as well, that is any differentiation is defined by a vector field, and its components are just the values of the differentiation on the coordinate functions x^i. The commutator of the differentiations defined as $[F,G](a) = F(G(a)) - G(F(a))$ corresponds to the commutator of the vector fields

$$[F,G] = (F(g^1) - G(f^1))\partial_{x^1} + \cdots + (F(g^n) - G(f^n))\partial_{x^n}$$

which equips the space of the vector fields with the structure of a Lie algebra.

The existence of some additional structures on the manifold allows to distinguish several special Lie subalgebras of the vector fields. For example, a Hamiltonian vector field X_H is defined by a single function H accordingly to the rule $X_H(a) = \{H, a\}$ if M is equipped with a Poisson bracket.

➢ See also: contact vector field, evolutionary derivative.

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218 Vector integrable evolutionary equations

Author: V.V. Sokolov, 08.02.2009

1. Introduction. Examples

An example of vector equation is given by two different vector analogs of mKdV equation:

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + (\boldsymbol{u}, \boldsymbol{u})\boldsymbol{u}_1, \qquad \text{and} \qquad \boldsymbol{u}_t = \boldsymbol{u}_3 + (\boldsymbol{u}, \boldsymbol{u})\boldsymbol{u}_1 + (\boldsymbol{u}, \boldsymbol{u}_1)\boldsymbol{u}.$$

Here $\boldsymbol{u}$ denotes a N-dimensional vector and $(\cdot, \cdot)$ stands for the standard scalar product. It is well known that these equation are integrable at any N by inverse scattering method and, consequently, possess infinite sets of symmetries and conservation laws. It is clear that both equations possess $SO(N)$-symmetry (that is, are invariant with respect to all rotations). Such equations are called ***isotropic***. These examples belong to the class of isotropic equations of the general form

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + f_2\boldsymbol{u}_2 + f_1\boldsymbol{u}_1 + f_0\boldsymbol{u} \tag{1}$$

where the coefficients f_i are real-valued functions with the argument set of six different scalar products of the vectors $\boldsymbol{u}, \boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

A more general class of equations (1) consists of vector ***anisotropic*** equations. The example of such an equation is [1]

$$\boldsymbol{u}_t = \Big(\boldsymbol{u}_2 + \frac{3}{2}(\boldsymbol{u}_1, \boldsymbol{u}_1)\boldsymbol{u}\Big)_x + \frac{3}{2}(\boldsymbol{u}, R\boldsymbol{u})\boldsymbol{u}_1, \qquad (\boldsymbol{u}, \boldsymbol{u}) = 1 \tag{2}$$

where R is an arbitrary constant symmetric matrix. If $N = 3$ then (2) is the symmetry of the famous Landau–Lifshitz equation. Equation (2) is integrable for any N and R. In contrast to the isotropic case, the coefficients of anisotropic equations (1) depend on six more arguments defined by means of additional scalar product $\langle X, Y\rangle = (X, RY)$.

2. Definitions and notations

Let us introduce the following notations. In the isotropic case, let $\mathcal{F}$ denote the set of locally analytic functions on the variables

$$u_{[i,j]} = (\boldsymbol{u}_i, \boldsymbol{u}_j), \quad 0 \le i \le j. \tag{3}$$

Here $(\cdot,\cdot)$ is a scalar product in a N-dimensional (or infinite-dimensional) vector space V.

In the anisotropic situation, let $\mathcal{F}$ denote the set of locally analytic functions on the variables (3) and additional variables

$$\tilde{u}_{[i,j]} = \langle \boldsymbol{u}_i, \boldsymbol{u}_j\rangle, \quad 0 \le i \le j \tag{4}$$

where $\langle\cdot,\cdot\rangle$ is another scalar product in V. The differential order of the variables $u_{[i,j]}$ and $\tilde{u}_{[i,j]}$ is equal j. If a function depends on variables (3) and (4) up to order n then we say that n is order of this function.

We consider the properties of vector equations which do not depend on the nature of the space V and of the scalar products. This important assumption is formalized as follows: no constraints exist between the scalar products (3), (4) which will play the role of *independent* variables.

The results on the formal symmetry obtained in papers [2, 3, 4, 5] can be generalized in the vector case as follows:

Theorem 1 ([6]). *i) If equation (1) possesses an infinite sequence of the symmetries of the form*

$$\boldsymbol{u}_\tau = g_m \boldsymbol{u}_m + g_{m-1}\boldsymbol{u}_{m-1} + \cdots + g_1\boldsymbol{u}_1 + g_0\boldsymbol{u}$$

then a formal series exists

$$L = a_1 D_x + a_0 + a_{-1}D_x^{-1} + a_{-2}D_x^{-2} + \cdots, \quad a_i \in \mathcal{F}$$

which satisfies the operator relation

$$L_t = [A, L], \quad A = D_x^3 + f_2 D_x^2 + f_1 D_x + f_0. \tag{5}$$

ii) The functions

$$\rho_{-1} = \frac{1}{a_1}, \quad \rho_0 = \frac{a_0}{a_1}, \quad \rho_i = \operatorname{res} L^i, \quad i \in \mathbb{N} \tag{6}$$

are conserved densities of equation (1).

iii) If equation (1) possesses an infinite sequence of conservation laws with the densities belonging to $\mathcal{F}$ then a formal symmetry L exists as well as the formal series S of the form

$$S = s_1 D_x + s_0 + s_{-1} D_x^{-1} + s_{-2} D_x^{-2} + \cdots \qquad s_i \in \mathcal{F}$$

such that

$$S_t + A^{\mathsf{T}} S + SA = 0, \quad S^{\mathsf{T}} = -S$$

where the superscript $^{\mathsf{T}}$ denotes the transposition in the algebra of formal series (see e.g. [5]).

iv) If the conditions of the part iii) are fulfilled then the canonical conservation laws (6) with $i = 2k$ are trivial (that is, the densities are of the form $\rho_{2k} = D_x(\sigma_k)$ with some functions $\sigma_k \in \mathcal{F}$).

Notice, that the operator A in the relation (5) is not the Frechet derivative of the right hand side of equation, in contrast to the scalar case.

The conservation laws

$$D_t \rho_n = D_x \theta_n, \quad n \geq 0$$

described in the Theorem 1 are called canonical. They can be defined by recurrent formula [6]:

$$\begin{aligned}\rho_{n+2} = {} & \frac{1}{3}\Big[\theta_n - f_0 \delta_{n,0} - 2 f_2 \rho_{n+1} - f_2 D_x \rho_n - f_1 \rho_n\Big] \\ & - \frac{1}{3}\Big[f_2 \sum_{s=0}^{n} \rho_s \rho_{n-s} + \sum_{0 \le s+k \le n} \rho_s \rho_k \rho_{n-s-k} + 3 \sum_{s=0}^{n+1} \rho_s \rho_{n-s+1}\Big] \\ & - D_x\Big[\rho_{n+1} + \frac{1}{2}\sum_{s=0}^{n} \rho_s \rho_{n-s} + \frac{1}{3} D_x(\rho_n)\Big], \quad n \geq 0 \end{aligned} \tag{7}$$

where $\delta_{i,j}$ is Kronecker delta, ρ_0 and ρ_1 are of the form

$$\rho_0 = -\frac{1}{3} f_2, \quad \rho_1 = \frac{1}{9} f_2^2 - \frac{1}{3} f_1 + \frac{1}{3} D_x(f_2).$$

In particular, the next conserved density can be found by this formula is

$$\rho_2 = -\frac{1}{3} f_0 + \frac{1}{3}\theta_0 - \frac{2}{81} f_2^3 + \frac{1}{9} f_1 f_2 - D_x \left(\frac{1}{9} f_2^2 + \frac{2}{9} D_x(f_2) - \frac{1}{3} f_1 \right).$$

3. Isotropic equations on the sphere

The constraint $(\boldsymbol{u}, \boldsymbol{u}) := u_{[0,0]} = 1$ appears on the sphere by definition, and it implies also $u_{[0,1]} = 0$, $u_{[0,2]} = -u_{[1,1]}$, $u_{[0,3]} = -3u_{[1,2]}$ and so on. These relations allow to eliminate $u_{[0,i]}$, $i = 0, 1, 2, \ldots$ from the set of independent scalar products.

Moreover, the condition $(\boldsymbol{u}, \boldsymbol{u}_t) = 0$ holds on the sphere which implies $f_0 = f_2 u_{[1,1]} + 3u_{[1,2]}$. Thus, the equation on the sphere takes the form

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + f_2\, \boldsymbol{u}_2 + f_1\, \boldsymbol{u}_1 + (f_2\, u_{[1,1]} + 3\, u_{[1,2]})\, \boldsymbol{u}. \tag{8}$$

In this section we consider isotropic equations on the sphere. We assume, without loss of generality, that coefficients f_1 and f_2 in equation (8) depend on the variables $u_{[1,1]}$, $u_{[1,2]}$, $u_{[2,2]}$ only.

The complete list of isotropic integrable equations on the sphere was obtained in [6]:

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\boldsymbol{u}_2 + \frac{3}{2}\frac{u_{[2,2]}}{u_{[1,1]}}\boldsymbol{u}_1, \tag{9}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\boldsymbol{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2(1 + a u_{[1,1]})}\right)\boldsymbol{u}_1, \tag{10}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + \frac{3}{2}\left(\frac{a^2 u_{[1,2]}^2}{1 + a u_{[1,1]}} - a(u_{[2,2]} - u_{[1,1]}^2) + u_{[1,1]}\right)\boldsymbol{u}_1 + 3u_{[1,2]}\boldsymbol{u}, \tag{11}$$

$$
\begin{aligned}
\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{(q+1)u_{[1,2]}}{2qu_{[1,1]}}\boldsymbol{u}_2 + 3\frac{(q-1)u_{[1,2]}}{2q}\boldsymbol{u} \\
+ \frac{3}{2}\left(\frac{(q+1)u_{[2,2]}}{u_{[1,1]}} - \frac{(q+1)au_{[1,2]}^2}{q^2u_{[1,1]}} + u_{[1,1]}(1-q)\right)\boldsymbol{u}_1
\end{aligned}
\tag{12}
$$

where a is an arbitrary constant and $q = \varepsilon\sqrt{1 + au_{[1,1]}}$, $\varepsilon^2 = 1$.

Remark 2. A more detailed list appears if we consider the cases $a = 0$ or $a \neq 0$ separately. In particular, equation (12) with $a = 0$ and $\varepsilon = -1$ is of the form

$$
\boldsymbol{u}_t = \boldsymbol{u}_3 + 3u_{[1,1]}\boldsymbol{u}_1 + 3u_{[1,2]}\boldsymbol{u}. \tag{13}
$$

If $a = 0$ and $\varepsilon = 1$ then equation (12) takes another form:

$$
\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\boldsymbol{u}_2 + 3\frac{u_{[2,2]}}{u_{[1,1]}}\boldsymbol{u}_1. \tag{14}
$$

Remark 3. Each equation from the list admits a fifth order symmetry. For instance, the symmetry of equation (13) is

$$
\boldsymbol{u}_\tau = \boldsymbol{u}_5 + 5u_{[1,1]}\boldsymbol{u}_3 + 15u_{[1,2]}\boldsymbol{u}_2 + 5\left(3u_{[1,1]}^2 + 2u_{[2,2]} + 3u_{[1,3]}\right)\boldsymbol{u}_1 + 5\left(6u_{[1,2]}u_{[1,1]} + 2u_{[2,3]} + u_{[1,4]}\right)\boldsymbol{u}.
$$

Remark 4. Equation (9) in $\mathbb{R}^N$ appeared in the papers [7, 8, 9] in connection with triple Jordan systems. This is the vector analog of well known Schwarz–KdV equation.

Remark 5. Equations (10) and (11) $a = 0$ on the one-dimensional sphere are reduced to the potential KdV equation

$$
v_t = v_{xxx} + v_x^3
$$

by use of stereographic projection and certain point transformations. In the case $a = -1$, these equations are reduced to Calogero–Degasperis equation

$$
u_t = u_{xxx} - \frac{1}{2}Q''u_x + \frac{3}{8}\frac{((Q - u_x^2)_x)^2}{u_x(Q - u_x^2)}
$$

where $Q(v) = \frac{1}{4}(v^2 + 1)^2$. This form of polynomial $Q(v)$ corresponds to the trigonometric degeneration of the elliptic curve underlying the Calogero–Degasperis equation.

Equation (12) is reduced to integrable equation

$$v_t = v_{xxx} - \frac{6av_x v_{xx}^2}{1 + 4av_x^2} + 8v_x^3.$$

4. Anisotropic equations on the sphere

Coefficients of anisotropic equations (1) on the sphere depend a priori on nine variables. The complete list of integrable equations was obtained in [6, 10] (a and b are arbitrary constants):

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + \frac{3}{2}\Big(u_{[1,1]} + \tilde{u}_{[0,0]}\Big)\boldsymbol{u}_1 + 3u_{[1,2]}\boldsymbol{u}, \tag{15}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\boldsymbol{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2} + \frac{\tilde{u}_{[1,1]}}{u_{[1,1]}}\right)\boldsymbol{u}_1, \tag{16}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\boldsymbol{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2} - \frac{(\tilde{u}_{[0,1]} + u_{[1,2]})^2}{(u_{[1,1]} + \tilde{u}_{[0,0]})u_{[1,1]}} + \frac{\tilde{u}_{[1,1]}}{u_{[1,1]}}\right)\boldsymbol{u}_1, \tag{17}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{\tilde{u}_{[0,1]}}{\tilde{u}_{[0,0]}}\boldsymbol{u}_2 - 3\left(\frac{2\tilde{u}_{[0,2]} + \tilde{u}_{[1,1]} + a}{2\tilde{u}_{[0,0]}} - \frac{5}{2}\frac{\tilde{u}_{[0,1]}^2}{\tilde{u}_{[0,0]}^2}\right)\boldsymbol{u}_1 + 3\left(u_{[1,2]} - \frac{\tilde{u}_{[0,1]}}{\tilde{u}_{[0,0]}}u_{[1,1]}\right)\boldsymbol{u}, \tag{18}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{\tilde{u}_{[0,1]}}{\tilde{u}_{[0,0]}}\boldsymbol{u}_2 - 3\left(\frac{\tilde{u}_{[0,2]}}{\tilde{u}_{[0,0]}} - 2\frac{\tilde{u}_{[0,1]}^2}{\tilde{u}_{[0,0]}^2}\right)\boldsymbol{u}_1 + 3\left(u_{[1,2]} - \frac{\tilde{u}_{[0,1]}}{\tilde{u}_{[0,0]}}u_{[1,1]}\right)\boldsymbol{u}, \tag{19}$$

$$\begin{aligned}\boldsymbol{u}_t = \boldsymbol{u}_3 &- 3\frac{\tilde{u}_{[0,1]}}{\tilde{u}_{[0,0]}}\left(\boldsymbol{u}_2 + u_{[1,1]}\boldsymbol{u}\right) + 3u_{[1,2]}\boldsymbol{u} \\ &+ \frac{3}{2}\left(-\frac{u_{[2,2]}}{\tilde{u}_{[0,0]}} + \frac{(u_{[1,2]} + \tilde{u}_{[0,1]})^2}{\tilde{u}_{[0,0]}(\tilde{u}_{[0,0]} + u_{[1,1]})} + \frac{(\tilde{u}_{[0,0]} + u_{[1,1]})^2}{\tilde{u}_{[0,0]}} + \frac{\tilde{u}_{[0,1]}^2 - \tilde{u}_{[0,0]}\tilde{u}_{[1,1]}}{\tilde{u}_{[0,0]}^2}\right)\boldsymbol{u}_1,\end{aligned} \tag{20}$$

$$
\begin{aligned}
\boldsymbol{u}_t &= \boldsymbol{u}_3 + 3\left(\frac{\tilde u_{[0,1]}\tilde u_{[0,2]}}{\xi} - \frac{\tilde u_{[1,2]}\tilde u_{[0,0]}}{\xi} + \frac{\tilde u_{[0,1]}}{\tilde u_{[0,0]}}\right)\big(\boldsymbol{u}_2 + u_{[1,1]}\boldsymbol{u}\big) + 3u_{[1,2]}\boldsymbol{u} \\
&\quad + \frac{3}{2\xi^2\tilde u_{[0,0]}^2}\Big(\tilde u_{[0,0]}^3\tilde u_{[2,2]}\xi - \xi(\xi + \tilde u_{[0,2]}\tilde u_{[0,0]})^2 + (\tilde u_{[0,0]}^2\tilde u_{[1,2]} - 2\xi\tilde u_{[0,1]} - \tilde u_{[0,0]}\tilde u_{[0,1]}\tilde u_{[0,2]})^2\Big)\boldsymbol{u}_1 \\
&\quad - a\frac{\tilde u_{[0,0]}^2 u_{[1,1]} + \tilde u_{[0,1]}^2}{\tilde u_{[0,0]}\xi}\boldsymbol{u}_1, \qquad \xi = \tilde u_{[0,0]}\tilde u_{[1,1]} - \tilde u_{[0,1]}^2,
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\boldsymbol{u}_t &= \boldsymbol{u}_3 + 3\left(\frac{\tilde u_{[0,1]}\tilde u_{[0,2]}}{\xi} - \frac{\tilde u_{[1,2]}\tilde u_{[0,0]}}{\xi} + \frac{\tilde u_{[0,1]}}{\tilde u_{[0,0]}}\right)\big(\boldsymbol{u}_2 + u_{[1,1]}\boldsymbol{u}\big) + 3u_{[1,2]}\boldsymbol{u} \\
&\quad + \frac{3}{\xi}\left(\tilde u_{[0,0]}\tilde u_{[2,2]} - 2\tilde u_{[0,1]}\tilde u_{[1,2]} - \frac{(\tilde u_{[0,2]}\tilde u_{[0,0]} - 2\tilde u_{[0,1]}^2)(\xi + \tilde u_{[0,2]}\tilde u_{[0,0]})}{\tilde u_{[0,0]}^2}\right)\boldsymbol{u}_1, \\
&\qquad \xi = \tilde u_{[0,0]}\tilde u_{[1,1]} - \tilde u_{[0,1]}^2,
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\boldsymbol{u}_t &= \boldsymbol{u}_3 - 3\frac{a\tilde u_{[0,1]}}{\eta}\boldsymbol{u}_2 + 3\frac{u_{[1,2]}\eta - a\tilde u_{[0,1]}u_{[1,1]}}{\eta}\boldsymbol{u} + \frac{3}{2}\left(\frac{\tilde u_{[2,2]}}{\eta} + \frac{a\xi - (\tilde u_{[0,2]} + \eta)^2}{\eta\tilde u_{[0,0]}}\right)\boldsymbol{u}_1 \\
&\quad + \frac{3}{2}\left(\frac{\big(\tilde u_{[0,0]}\tilde u_{[1,2]} - \tilde u_{[0,1]}(2a\tilde u_{[0,0]} + b + \tilde u_{[0,2]})\big)^2}{\eta\xi\tilde u_{[0,0]}} - b\frac{a\tilde u_{[0,1]}^2 + \eta\tilde u_{[0,0]}u_{[1,1]}}{\eta^2\tilde u_{[0,0]}}\right)\boldsymbol{u}_1, \\
&\qquad \eta = a\tilde u_{[0,0]} + b, \qquad \xi = \tilde u_{[0,0]}(\eta - \tilde u_{[1,1]}) + \tilde u_{[0,1]}^2,
\end{aligned} \tag{23}
$$

$$
\boldsymbol{u}_t = \boldsymbol{u}_3 + 3\left(\frac{\tilde u_{[0,1]}}{\tilde u_{[0,0]}} + \frac{\tilde u_{[0,1]}\tilde u_{[0,2]}}{\xi} - \frac{\tilde u_{[0,0]}\tilde u_{[1,2]}}{\xi}\right)\big(\boldsymbol{u}_2 + u_{[1,1]}\boldsymbol{u}\big) + 3u_{[1,2]}\boldsymbol{u}
$$

$$+\frac{3}{2}\left(\frac{\tilde u_{[0,0]}\tilde u_{[2,2]}}{\xi}+b\tilde u_{[0,0]}\frac{u_{[1,1]}\eta+a\tilde u^2_{[0,1]}}{\eta\xi}-\frac{(\tilde u_{[0,0]}\tilde u_{[0,2]}+\xi)^2}{\tilde u^2_{[0,0]}\xi}\right)\boldsymbol{u}_1$$

$$+\frac{3}{2}\frac{\big(\tilde u^2_{[0,0]}(a\xi\tilde u_{[0,1]}-\eta\tilde u_{[1,2]})+\eta\tilde u_{[0,1]}(\tilde u_{[0,0]}\tilde u_{[0,2]}+\xi)\big)^2}{\eta\tilde u^2_{[0,0]}(\xi+\eta)\xi^2}\boldsymbol{u}_1,$$

$$\xi=\tilde u_{[0,0]}\tilde u_{[1,1]}-\tilde u^2_{[0,1]},\quad \eta=(a\tilde u_{[0,0]}+b)\tilde u_{[0,0]}, \tag{24}$$

$$\boldsymbol{u}_t=\boldsymbol{u}_3+\frac{3}{2}\left(\frac{\tilde u_{[0,0]}\tilde u_{[1,2]}-\tilde u_{[0,1]}\tilde u_{[0,2]}}{\mu(\mu+\tilde u_{[0,0]})}-2\frac{\tilde u_{[0,1]}}{\mu}\right)(\boldsymbol{u}_2+u_{[1,1]}\boldsymbol{u})+3u_{[1,2]}\boldsymbol{u}$$

$$+\frac{3}{2\tilde u_{[0,0]}(\mu+\tilde u_{[0,0]})}\Big[\mu^{-2}\big(\tilde u_{[0,0]}\tilde u_{[1,2]}-\tilde u_{[0,1]}\tilde u_{[0,2]}\big)^2+\tilde u_{[0,0]}\tilde u_{[2,2]}-\tilde u^2_{[0,2]}$$

$$-2\mu^{-2}\tilde u_{[0,1]}\big(\tilde u_{[0,0]}\tilde u_{[1,2]}-\tilde u_{[0,1]}\tilde u_{[0,2]}\big)(\mu+2\tilde u_{[0,0]})\Big]\boldsymbol{u}_1$$

$$+\big(6\mu^{-2}\tilde u^2_{[0,1]}-3\tilde u^{-1}_{[0,0]}\tilde u_{[0,2]}\big)\boldsymbol{u}_1,\quad \mu^2=\tilde u^2_{[0,1]}+\tilde u^2_{[0,0]}-\tilde u_{[0,0]}\tilde u_{[1,1]}. \tag{25}$$

Remark 6. Equations (15) – (17) were announced in [6]. Equation (15) coincides with (2).

Remark 7. The presented list can be considered in more details. For instance, one can assume $a=0$ in (18) and (21). It is possible to assume $a=0$ or $b=0$, but $\{a,b\}\neq 0$ in (23). Equation (23) takes the following form at $a=0$:

$$\boldsymbol{u}_t=\boldsymbol{u}_3+\frac{3}{2}\left(\frac{\tilde u_{[2,2]}}{b}-\frac{(\tilde u_{[0,2]}+b)^2}{b\tilde u_{[0,0]}}+\frac{\big(\tilde u_{[0,0]}\tilde u_{[1,2]}-\tilde u_{[0,1]}(\tilde u_{[0,2]}+b)\big)^2}{b\xi\tilde u_{[0,0]}}-u_{[1,1]}\right)\boldsymbol{u}_1+3\,u_{[1,2]}\boldsymbol{u} \tag{23a}$$

where $\xi=\tilde u_{[0,0]}(b-\tilde u_{[1,1]})+\tilde u^2_{[0,1]}$.

The assumptions $a = 0$ and then $b = 0$ in (24) reduce this equation to the form

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + 3\left(\frac{\tilde u_{[0,1]}}{\tilde u_{[0,0]}} + \frac{\tilde u_{[0,1]}\tilde u_{[0,2]} - \tilde u_{[0,0]}\tilde u_{[1,2]}}{\xi}\right)(\boldsymbol{u}_2 + u_{[1,1]}\boldsymbol{u}) + 3u_{[1,2]}\boldsymbol{u}$$

$$+\frac{3}{2}\left(\frac{\tilde u_{[0,0]}\tilde u_{[2,2]}}{\xi} - \frac{(\xi + \tilde u_{[0,0]}\tilde u_{[0,2]})^2}{\xi \tilde u_{[0,0]}^2}\right)\boldsymbol{u}_1, \quad \xi = \tilde u_{[0,0]}\tilde u_{[1,1]} - \tilde u_{[0,1]}^2. \tag{24a}$$

This is an anisotropic generalization of Schwarz-KdV equation.

5. Bäcklund transformations for equations on the sphere

All integrable equations on the sphere admits auto-Bäcklund transformations. These transformations contain an arbitrary parameter which allows, in principle, to construct multisoliton and finite-gap solutions, even if the Lax representation is not known (see [11])

The first order Bäcklund auto-transformation for a scalar evolutionary equation is a relation between two solutions u and v of this equation and their derivatives u_x and v_x. In the vector case, first order Bäcklund auto-transformation were introduced in the paper [6] as ODE of the form

$$\boldsymbol{u}_1 = f\boldsymbol{u} + g\boldsymbol{v} + h\boldsymbol{v}_x \tag{26}$$

where f, g and h are certain scalar function depending on the products of the vectors $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{v}_x$. The arguments of f, g and h, in the case of isotropic equation in $\mathbb{R}^n$, are

$$u_{[0,0]} = (\boldsymbol{u},\boldsymbol{u}), \quad w_0 = (\boldsymbol{u},\boldsymbol{v}), \quad w_1 = (\boldsymbol{u},\boldsymbol{v}_x), \quad v_{[0,0]} = (\boldsymbol{v},\boldsymbol{v}), \quad v_{[0,1]} = (\boldsymbol{v},\boldsymbol{v}_x), \quad v_{[1,1]} = (\boldsymbol{v}_x,\boldsymbol{v}_x).$$

In the anisotropic case, the products

$$\tilde u_{[0,0]} = \langle\boldsymbol{u},\boldsymbol{u}\rangle, \quad \tilde w_0 = \langle\boldsymbol{u},\boldsymbol{v}\rangle, \quad \tilde w_1 = \langle\boldsymbol{u},\boldsymbol{v}_x\rangle, \quad \tilde v_{[0,0]} = \langle\boldsymbol{v},\boldsymbol{v}\rangle, \quad \tilde v_{[0,1]} = \langle\boldsymbol{v},\boldsymbol{v}_x\rangle, \quad \tilde v_{[1,1]} = \langle\boldsymbol{v}_x,\boldsymbol{v}_x\rangle$$

should be added. The constraints eliminate the variables $u_{[0,0]}, v_{[0,0]}$ and $v_{[0,1]}$ in the case of equations on the sphere or on a cone.

In order to find Bäcklund auto-transformation for the evolutionary equation (1) we differentiate (26) with respect to t in virtue of equation (1) and then eliminate $\boldsymbol{u}_1$ by use of (26). By the definition of Bäcklund transformation, the resulting equation must hold identically. The splitting of this equation with respect to those independent variables which do not occur as arguments of the functions f, g and h brings to an overdetermined system on nonlinear PDE for these functions. If this system has a solution which depend essentially on the parameter λ then this solution defines the desired Bäcklund auto-transformation.

6. Divergent equations

The general classification problem for integrable equations (1) is very cumbersome and it is not solved at present. The coefficients of the equation depend on the large number of variables, but this is not the only difficulty. The known examples (see [12, 13]) demonstrate that the dependence of these coefficients on their arguments can be extremely complicated.

The problem which leads to a quite visible answer is the classification of integrable vector evolutionary equations of the form

$$\boldsymbol{u}_t = (\boldsymbol{u}_2 + f_1\boldsymbol{u}_1 + f_0\boldsymbol{u})_x, \qquad f_i = f_i(u_{[0,0]}, \tilde{u}_{[0,0]}, u_{[0,1]}, \tilde{u}_{[0,1]}, u_{[1,1]}, \tilde{u}_{[1,1]})$$

where f_i are scalar functions. The list of such equations, obtained in [14] is as follows, after the transformation to the potential form by the change $\boldsymbol{u} \to \boldsymbol{u}_1$:

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + \frac{3}{2}\Big(\frac{a^2 {u_{[1,2]}}^2}{1 + au_{[1,1]}} - au_{[2,2]}\Big)\boldsymbol{u}_1, \tag{27}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\boldsymbol{u}_2 + \frac{3u_{[2,2]}}{3u_{[1,1]}}\boldsymbol{u}_1, \tag{28}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\boldsymbol{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{{u_{[1,2]}}^2}{{u_{[1,1]}}^2(1 + au_{[1,1]})}\right)\boldsymbol{u}_1, \tag{29}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - \frac{3}{2}(p+1)\frac{u_{[1,2]}}{pu_{[1,1]}}\boldsymbol{u}_2 + \frac{3}{2}(p+1)\left(\frac{u_{[2,2]}}{u_{[1,1]}} - \frac{a{u_{[1,2]}}^2}{p^2 u_{[1,1]}}\right)\boldsymbol{u}_1, \tag{30}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\boldsymbol{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{{u_{[1,2]}}^2}{{u_{[1,1]}}^2} + a\frac{\tilde{u}_{[1,1]}}{u_{[1,1]}}\right)\boldsymbol{u}_1, \tag{31}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\boldsymbol{u}_2 + 3\frac{u_{[2,2]}}{u_{[1,1]}}\boldsymbol{u}_1. \tag{32}$$

Here $p = \sqrt{1 + au_{[1,1]}}$ and a is a constant.

It should be noted that the form of equations (28), (29), (31) and (32) coincide with the respective equations on the sphere.

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219 Veselov–Novikov equation

$$u_t = \alpha(u_{xx} + 3p_x u)_x + \beta(u_{yy} + 3q_y u)_y, \quad p_y = u, \quad q_x = u$$

Alias: BKP

➢ Linear problem:

$$\psi_{xy} = u\psi, \quad \psi_t = \alpha(\psi_{xxx} + 3p_x\psi_x) + \beta(\psi_{yyy} + 3q_y\psi_y).$$

➢ VN equation appears as the reduction $v = 1$ of the 3-rd order symmetry (40.1) of the Davey–Stewartson system.

➢ Higher symmetry:

$$u_{t_5} = (u_{xxxx} + 5(u_x w_x)_x + 5u(w_{xxx} + w_x^2 + w_{1,x}))_x, \quad w_y = u, \quad w_{1,y} = uw_x.$$

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220 Veselov–Novikov equation modified

$$u_t = u_{xxx} + 3u_x w_x + \frac{3}{2} u w_{xx}, \quad w_y = u^2$$

This is the reduction $v = u$ of the 3-rd order symmetry (40.1) of the Davey–Stewartson system.

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221 Volterra lattice

$$u_{n,x} = u_n(u_{n+1} - u_{n-1}) \qquad (1)$$

Aliases: Lotka–Volterra model, Kac-van Moerbeke lattice, Langmuir lattice.

➢ Bi-Hamiltonian structure [5, 6, 7]:

$$u_{n,x} = \{u_n, H^{(1)}\}_1 = \{u_n, H^{(2)}\}_2,$$

$$\{u_n, u_{n+1}\}_1 = u_n u_{n+1}, \quad H^{(1)} = \sum u_n$$

$$\{u_n, u_{n+1}\}_2 = u_n u_{n+1}(u_n + u_{n+1}), \quad \{u_n, u_{n+2}\}_2 = u_n u_{n+1} u_{n+2}, \quad H^{(2)} = \frac{1}{2}\sum \log u_n$$

➢ Bäcklund transformation:

$$u_n = (f_n + \delta)(f_{n+1} - \delta), \quad \tilde{u}_n = (f_{n+1} + \delta)(f_n - \delta).$$

The variable f satisfies the modified Volterra lattice $f_{n,x} = (f_n^2 - \delta^2)(f_{n+1} - f_{n-1})$.

➢ Zero curvature representation:

$$U_n = \begin{pmatrix} u_n & -\lambda u_n \\ -\lambda & \lambda^2 + u_{n-1} \end{pmatrix}, \quad L_n = \begin{pmatrix} \lambda & u_n \\ 1 & 0 \end{pmatrix}, \quad M_n = \begin{pmatrix} -\dfrac{\lambda}{2\delta} & f_n + \delta \\ \dfrac{1}{f_n - \delta} & -\dfrac{\lambda}{f_n - \delta} - \dfrac{\lambda}{2\delta} \end{pmatrix}$$

➢ The lattice [8]

$$p_{n,x} = p_n(s_{n+1} - s_{n-1}), \quad -s_{n,x} = s_n(p_{n+1} - p_{n-1})$$

is splitted into two disjoint copies of the Volterra lattice (1) and $\tilde{u}_{n,x} = \tilde{u}_n(\tilde{u}_{n+1} - \tilde{u}_{n-1})$ after the change $u_{2n} = -p_{2n}$, $u_{2n+1} = s_{2n+1}$, $\tilde{u}_{2n} = s_{2n}$, $\tilde{u}_{2n+1} = -p_{2n+1}$.

➢ Nonabelian generalization: let A be an associative algebra with unity, then the lattice [9]

$$u_{n,x} = u_{n+1}u_n - u_n u_{n-1}, \quad u_n \in A$$

admits the ZCR with the matrices U_n, W_n of the same form as in scalar case. The simplest higher symmetry takes the form

$$u_{n,t} = u_{n+2}u_{n+1}u_n + u_{n+1}^2 u_n + u_{n+1}u_n^2 - u_n^2 u_{n-1} - u_n u_{n-1}^2 - u_n u_{n-1} u_{n-2}.$$

➢ Another multifield generalization [10, 11]:

$$u_{n,x}^{(j)} = u_n^{(j)}\Big(\sum_{k=1}^{j-1}(u_{n+1}^{(k)} - u_n^{(k)}) - \sum_{k=j+1}^{m}(u_n^{(k)} - u_{n-1}^{(k)})\Big), \quad j = 1, \dots, m$$

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222 Volterra lattice modified

$$u_{n,x} = (1 - u_n^2)(u_{n+1} - u_{n-1})$$

Alias: discrete mKdV equation.

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223 Volterra lattice twodimensional

The version from [1, 2]:

$$u_t = u(u_{-1}^2 - u_1^2) \pm w_y, \quad (uu_{-1})_y = uw_{-1} - u_{-1}w$$

The version from [3, 4]:

$$u_x = u(v - v_1), \quad v_y = v(u - u_{-1})$$

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224 Volterra type lattices, classification

Volterra-type lattices are differential-difference equations of the form (for short, let $u_n = u$, $u_{n\pm1} = u_{\pm1}$)

$$\dot{u} = f(u_1, u, u_{-1}). \tag{1}$$

They are named after the Volterra lattice which is one of the most important integrable models.

According to the general results of the symmetry approach, the existence of higher symmetries implies solvability of the necessary integrability conditions in the form of the conservation laws

$$D_t(\rho^{(j)}) = (T-1)(\sigma^{(j)}), \quad j = 0, 1, 2, \dots, \tag{2}$$

while the existence of higher order conservation laws implies conditions of the form

$$\hat{\rho}^{(j)} = (T-1)(\hat{\sigma}^{(j)}), \quad j = 0, 1, 2, \dots. \tag{3}$$

The quantities $\rho^{(j)}, \hat{\rho}^{(j)}$ are expressed explicitly in terms of the lattice r.h.s. and previously found $\sigma^{(j)}, \hat{\sigma}^{(j)}$, in particular

$$\begin{gathered} \rho^{(0)} = \log f_{u_1}, \quad \rho^{(1)} = f_u + \sigma^{(0)}, \quad \rho^{(2)} = f_{u_{-1}} T^{-1}(f_{u_1}) + \frac{1}{2}(\rho^{(1)})^2 + \sigma^{(1)}, \\ \hat{\rho}^{(0)} = \log(-f_{u_1}/f_{u_{-1}}), \quad \hat{\rho}^{(1)} = 2f_u + D_t(\hat{\sigma}^{(0)}). \end{gathered} \tag{4}$$

More precisely, $\rho^{(j)}$ are obtained by computing the free terms of the formal power series L^j where L is the formal symmetry defined by equation

$$D_t(L) = [f_*, L], \quad f_* := f_{u_1} T + f_u + f_{u_{-1}} T^{-1}, \quad L = a_1 T + a_0 + a_{-1} T^{-1} + \dots$$

and the conditions of the second kind are obtained from the formal conservation law

$$S_t + S f_* + f_*^{\mathsf{T}} S = 0, \quad S = s_0 + s_{-1} T^{-1} + s_{-2} T^{-2} + \dots$$

It turns out that few first conditions suffice to obtain the complete classification.

Theorem 1 (Yamilov [1]). *The lattices (1) satisfying the necessary integrability conditions (2), (3) with the quantities given in (4) are exhausted, up to the point transformations $\tilde{u} = a(u)$, by the following list:*

$$\dot{u} = P(u)(u_1 - u_{-1}) \tag{5}$$

$$\dot{u} = P(u^2)\Big(\frac{1}{u_1 + u} - \frac{1}{u + u_{-1}}\Big) \tag{6}$$

$$\dot{u} = Q(u)\Big(\frac{1}{u_1 - u} + \frac{1}{u - u_{-1}}\Big) \tag{7}$$

$$\dot{u} = \frac{H(u_1, u, u_{-1}) + \nu(H(u_1, u, u_1)H(u_{-1}, u, u_{-1}))^{1/2}}{u_1 - u_{-1}}, \quad \nu = 0, \pm 1 \tag{8}$$

$$\dot{u} = f(u_1 - u) + f(u - u_{-1}), \quad f' = P(f) \tag{9}$$

$$\dot{u} = f(u_1 - u)f(u - u_{-1}) + \mu, \quad f' = P(f)/f \tag{10}$$

$$\dot{u} = (f(u_1 - u) + f(u - u_{-1}))^{-1} + \mu, \quad f' = P(f^2) \tag{11}$$

$$\dot{u} = (f(u_1 + u) - f(u + u_{-1}))^{-1}, \quad f' = Q(f) \tag{12}$$

$$\dot{u} = \frac{f(u_1 + u) - f(u + u_{-1})}{f(u_1 + u) + f(u + u_{-1})}, \quad f' = P(f^2)/f \tag{13}$$

$$\dot{u} = \frac{f(u_1 + u) + f(u + u_{-1})}{f(u_1 + u) - f(u + u_{-1})}, \quad f' = Q(f)/f \tag{14}$$

$$\dot{u} = \frac{(1 - f(u_1 - u))(1 - f(u - u_{-1}))}{f(u_1 - u) + f(u - u_{-1})} + \mu, \quad f' = \frac{P(f^2)}{1 - f^2} \tag{15}$$

where $P''' = Q^V = 0$ and $H(u, v, w) = (\alpha v^2 + 2\beta v + \gamma)uw + (\beta v^2 + \lambda v + \delta)(u + w) + \gamma v^2 + 2\delta v + \varepsilon$.

All these lattices are integrable indeed, that is they belong to infinite hierarchies of commuting flows.

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225 Wadati–Konno–Ichikawa–Shimizu equation

$$iu_t = ((1 + u\bar{u})^{-1/2})_{xx}$$

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226 Wojciechowski system

$$\ddot{q}_k = \dot{p}_k = -\omega_k q_k + \frac{\mu_k^2}{q_k^3} - 2q_k \sum_{j=1}^{N} q_j^2, \quad k = 1, \ldots, N$$

➢ The Hamiltonian structure:

$$\{q_j, q_k\} = \{p_j, p_k\} = 0, \quad \{p_j, q_k\} = \delta_{jk}, \quad H = \frac{1}{2}\sum_{k=1}^{N}\left(p_k^2 + \omega_k q_k^2 + \frac{\mu_k^2}{q_k^2}\right) + \frac{1}{2}\left(\sum_{k=1}^{N} q_k^2\right)^2.$$

➢ The N independent first integrals in involution (assuming $\omega_k \neq \omega_j$, $\forall k, j$):

$$F_k = p_k^2 + \omega_k q_k^2 + \frac{\mu_k^2}{q_k^2} + q_k^2 \sum_{j=1}^{N} q_j^2 + \sum_{j \neq k} \frac{1}{\omega_j - \omega_k}\left((p_k q_j - p_j q_k)^2 + \frac{\mu_k^2 q_j^2}{q_k^2} + \frac{\mu_j^2 q_k^2}{q_j^2}\right), \quad F_1 + \cdots + F_N = 2H.$$

➢ The Lax pair $\dot{L} = [M, L]$:

$$L = \begin{pmatrix} \frac{1}{2}\lambda^2 I + \Omega + qq^\intercal & \lambda q + p + i\frac{\mu}{q} \\ -\lambda q^\intercal + p^\intercal - i\left(\frac{\mu}{q}\right)^\intercal & -\frac{1}{2}\lambda^2 - q^\intercal q \end{pmatrix}, \qquad M = \begin{pmatrix} -\frac{1}{2}\lambda I + i\dfrac{\mu}{q^2} & -q \\ q^\intercal & \frac{1}{2}\lambda \end{pmatrix}$$

where $p, q, \dfrac{\mu}{q}$ are column vectors with the k-th entry p_k, q_k, $\dfrac{\mu_k}{q_k}$ respectively and $\Omega = \operatorname{diag}(\omega_1, \ldots, \omega_N)$, $\dfrac{\mu}{q^2} = \operatorname{diag}\left(\dfrac{\mu_1}{q_1^2}, \ldots, \dfrac{\mu_N}{q_N^2}\right)$.

➢ See also: Rosochatius system

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227 Wronskian

Wide classes of explicit solutions to soliton equations, including rational, multi-soliton, multi-kink and others, possess a compact representation in terms of determinants. Each entry of such determinant is given by a simple expression corresponding in some way to a linear wave while the size of determinant depends on the number of poles of rational solution or the number of solitons. There are several types of such formulas related with Wronsky, Gram or Casorati type determinants or Pfaffians (recall that the Pfaffian of skew-symmetric matrix A of even order satisfies the relation $\mathrm{Pf}(A)^2 = \det(A)$).

Pfaffianization is a certain procedure which allows to replace multi-soliton solutions represented by determinants with solutions represented by Pfaffians, in expense of adding some extra field variables into the system under scrutiny. This procedure was originally applied to Kadomtsev–Petviashvili equation, resulting in Hirota–Ohta system [3]. Later on, this procedure was applied to many other equations, see e.g. [4].

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228 Yang–Baxter mappings

Author: V.E. Adler, 21.07.2005

Yang–Baxter maps, or *set-theoretical solutions of Yang–Baxter equation* [1] are 2D discrete equations which satisfy the property of 3D-consistency. A difference with the quad-equations is that the field variables are associated with the edges of square lattice rather than the vertices.

1. 3*D*-consistency

Consider mappings $R_{ij} : C_i \times C_j \to C_i \times C_j$ where C_i are some spaces or manifolds. Let the mapping $\hat{R}_{ij} : C_1 \times C_2 \times C_3 \to C_1 \times C_2 \times C_3$ act as R_{ij} on i-th and j-th factors and be identical on the rest one.

Definition 1. R_{ij} are called ***Yang–Baxter mappings*** if

$$\hat{R}_{23} \circ \hat{R}_{13} \circ \hat{R}_{12} = \hat{R}_{12} \circ \hat{R}_{13} \circ \hat{R}_{23}$$

We will use also an alternative definition. Let $F_{ij} : C_i \times C_j \to C_i \times C_j$ be given, in components, as

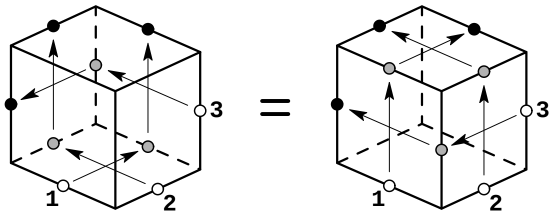
$$F_{ij} : (x^i, x^j) \mapsto (x^i_j, x^j_i) = (f^i_j(x^i, x^j), f^j_i(x^j, x^i)), \quad i, j = 1, 2, 3, \quad i \neq j.$$

Definition 2. The mappings F_{ij} are called ***3D-consistent*** if $x^i_{jk} \equiv x^i_{kj}$, that is

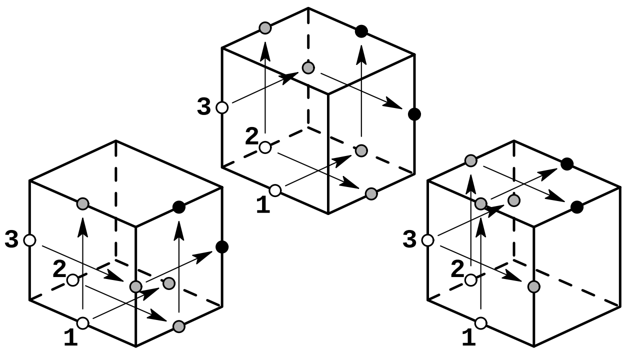
$$f^i_j(f^i_k(x^i, x^k), f^j_k(x^j, x^k)) = f^i_k(f^i_j(x^i, x^j), f^k_j(x^k, x^j)), \quad i \neq j \neq k \neq i.$$

Both notions define essentially the same property of consistency around the cube for mappings with variables on edges of the square lattice. These notions are equivalent under assumption that the mapping can be resolved w.r.t. variables on any adjacent pair of edges. In such situation, the difference is only in the order of computations and the choice of initial data, as shown on the following pictures (white, grey and black mark correspondingly initial data, intermediate values and consistency conditions).

Yang–Baxter mappings:



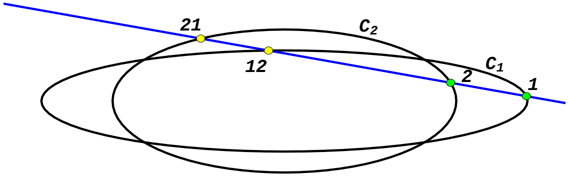
3D-consistent mappings:



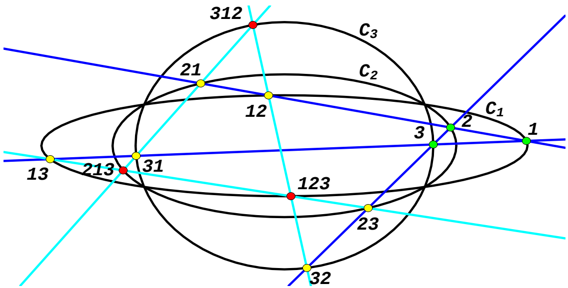
2. Yang–Baxter mappings on the linear pencils of conics

Let X^1, X^2 be points on conic sections C_1, C_2, respectively. Define the mapping $F_{12} : C_1 \times C_2 \to C_1 \times C_2$ as follows:

$$X^1_2 = X_1 X_2 \cap C_1, \quad X^2_1 = X_1 X_2 \cap C_2.$$



Now let us consider the initial data on three conics from the linear pencil. On the first step we apply the mappings $F_{ij} : (X_i, X_j) \mapsto (X^i_j, X^j_i)$. Next, we apply the mappings once more and see the remarkable incident theorem.



Theorem 3. *The mappings F_{ij} are 3D-consistent: $X^i_{jk} = X^i_{kj}$.*

Under a rational parametrization of the conics $C_i : X^i = X^i(x^i)$ the mapping F_{12} turns into a birational mapping on $\mathbb{CP}^1 \times \mathbb{CP}^1$. There exist 5 projective types of the linear pencils of conics $C_i = C + a_i K$ [2]. These types lead to the following list of the mappings ($i, j \in \{1, 2\}$):

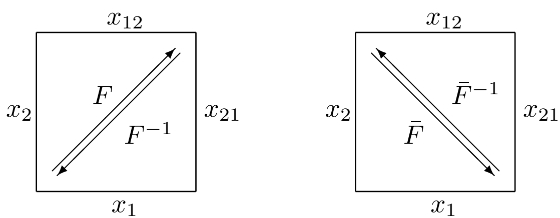
$$\begin{aligned}
x^i_j &= a_i x^j \frac{(1-a_2)x_1 + a_2 - a_1 + (a_1 - 1)x^2}{a_2(1-a_1)x^1 + (a_1 - a_2)x^2 x^1 + a_1(a_2 - 1)x^2} \\
x^i_j &= \frac{x^j}{a_i} \cdot \frac{a_1 x^1 - a_2 x^2 + a_2 - a_1}{x^1 - x^2} \\
x^i_j &= \frac{x^j}{a_i} \cdot \frac{a_1 x^1 - a_2 x^2}{x^1 - x^2} \\
x^i_j &= x^j \left(1 + \frac{a_2 - a_1}{x^1 - x^2}\right) \\
x^i_j &= x^j + \frac{a_1 - a_2}{x^1 - x^2}
\end{aligned} \tag{1}$$

The first one corresponds to the above figures with 4-point locus.

All these mappings can be obtained from those quad-equations listed in Theorem 187.3, which are invariant with respect to the shift $u \to u + c$ or scaling $u \to cu$, by the changes $x^i = u_i - u$ or $x^i = u_i/u$.

3. Quadrirational mappings

Definition 4 ([3, 4])**.** The mapping $F : C_1 \times C_2 \to C_1 \times C_2$ is called quadrirational if it and the mappings $F(x_1, \cdot) : C_2 \to C_2$, $F(\cdot, x_2) : C_1 \to C_1$ for almost all $x_i \in C_i$ are birational isomorphisms.



In the case $C_1 = C_2 = \mathbb{CP}^1$, a quadrirational mapping is of the form

$$F: \quad x_{12} = f(x_1, x_2) = \frac{a(x_2)x_1 + b(x_2)}{c(x_2)x_1 + d(x_2)}, \quad x_{21} = g(x_1, x_2) = \frac{A(x_1)x_2 + B(x_1)}{C(x_1)x_2 + D(x_1)}$$

with some special coefficients, such that the mappings F^{-1}, $\bar{F}$,$\bar{F}^{-1}$ be of the same form.

Assuming the nondegeneracy conditions

$$f_{x_1}g_{x_2} - f_{x_2}g_{x_1} \not\equiv 0, \quad f_{x_1} \not\equiv 0, \quad f_{x_2} \not\equiv 0, \quad g_{x_1} \not\equiv 0, \quad g_{x_2} \not\equiv 0,$$

one can prove that the coefficients can be at most quadratic polynomials. Moreover, the mapping F is defined by the pair of polynomial equations

$$P(x_2, x_1, x_{21}) = 0, \quad Q(x_2, x_{12}, x_{21}) = 0,$$

where either P, Q are linear in each argument or P, Q are linear in x_2, x_{21} and quadratic resp. in x_1, x_{12}, and are related by formula

$$Q(x_2, x_{12}, x_{21}) = (\gamma x_{12} + \delta)^2 P\Big(x_2, \frac{\alpha x_{12} + \beta}{\gamma x_{12} + \delta}, x_{21}\Big).$$

Theorem 5. *Up to the Möbius transformations, all nondegenerate quadrirational mappings, such that* $\max\deg(a, b, c, d) = \max\deg(A, B, C, D) = 2$*, are exhausted by the list (1).*

4. Multifield Yang–Baxter maps

The geometric construction of Yang–Baxter maps works also on the linear pencil of *quadrics*. Indeed, all points lie on the plane defined by the initial data X^1, X^2, X^3, so that $3D$-consistency is inherited from the planar situation. Nevertheless, the mapping itself cannot be reduced to the scalar one. Its general form is

$$X^i_j = X^j + \frac{(a_i - a_j)(\langle X^j, SX^j\rangle + \langle s, X^j\rangle + \sigma)}{\langle X^i - X^j, (a_iS + T)(X^i - X^j)\rangle}(X^i - X^j)$$

where S, T are arbitrary symmetric matrices, s is an arbitrary vector and σ is an arbitrary scalar.

Another examples of multifield Yang–Baxter maps were obtained in [5] by consideration of the interaction of matrix solitons with the non-trivial internal parameters (vector analog of phase shift).

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229 Yang–Mills equation

$$(U^{-1}U_{z_1})_{\bar{z}_1} + (U^{-1}U_{z_2})_{\bar{z}_2} = 0$$

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230 Zakharov system

$$i\psi_t = \psi_{xx} - n\psi, \quad n_t = u_x, \quad u_t = n_x + (|\psi|^2)_x$$

This nonintegrable system describes the nonlinear interaction of two waves corresponding to the different time-spatial scales.

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231 Zero curvature repesentation

A nonlinear equation admits the ***zero curvature representation*** (ZCR) if it is equivalent to the compatibility condition of a pair of auxiliary linear problems. Partial differential, differential-difference and difference-difference equations correspond to the following auxiliary problems:

$$
\begin{array}{llll}
\mathrm{DD}: & \Psi_x = U\Psi, & \Psi_t = V\Psi & \Rightarrow\ U_t = V_x + [V,U] \\
\mathrm{D}\Delta: & \Psi_{n,x} = U_n\Psi_n, & \Psi_{n+1} = L_n\Psi_n & \Rightarrow\ L_{n,x} = U_{n+1}L_n - L_nU_n \\
\Delta\Delta: & \Psi_{m,n+1} = L_{m,n}\Psi_{m,n}, & \Psi_{m+1,n} = M_{m,n}\Psi_{m,n} & \Rightarrow\ M_{m+1,n}L_{m,n} = L_{m,n+1}M_{m,n}
\end{array}
$$

where matrices depend on the variables of the equation, their derivatives or shifts, and the spectral parameter λ.

➢ A ZCR is called trivial, if it can be reduced to the scalar one or the spectral parameter can be eliminated.

➢ An important special case of ZCR is the Lax pair.

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232 3-wave equation

$$u_t = \alpha u_x + ivw^*, \quad v_t = \beta v_x + iuw, \quad w_t = \gamma w_x + iuv$$

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233 φ^4-equation

$$\varphi_{xx} - \varphi_{tt} = \pm(\varphi - \varphi^3)$$

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234 φ^6-equation

$$\varphi_{tt} = \Delta\varphi + c\varphi^5$$

Although this equation is not integrable, it possesses rich families of soliton-like solutions [1].

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